Efficiency of Semi–Analytic Technique for Solving Singular Eigen Value Problemswith Boundary Conditions

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Abstract

This paper proposes a semi-analytic technique to solve the second order singular eigenvalue problems for ordinary differential equation with boundary conditions using oscillator interpolation. The technique finds the eigenvalue and the corresponding nonzero eigenvector which represent the solution of the equation in a certain domain. Illustrative examples are presented, which confirm the theoretical predictions provided to demonstrate the efficiency and accuracy of the proposed method where the suggested solution compared with other methods.Also, proposed a new formula developed to estimate the error help reduce the accounts process and show the results are improved. The existing bvp code suite designed for the solution of boundary value problems was extended with a module for the computation of eigenvalues and eigen functions.

Key words:Eigenvalue problems, Singular eigenvalue problems.

1. Introduction

The eigenvalue problems (EVP′s) involves finding an unknown coefficient (eigenvalue) λ and the corresponding nonzero eigenvector that satisfy the solution of the problem (1).

The eigenvalue problems can be used in a variety of problems in science and engineering. For example, quadratic eigenvalue problems arise in an oscillation analysis with damping(2), (3)and stabilityproblems in fluid dynamics (4), and the three-dimensional (3D)Schrödinger equationcan result in a cubic eigenvalue problem (5).

Similarly, the study of higher order systems of differential equations leads to a matrix polynomial of degree greater than one (5). However, its applications are more complicated than standard and generalized eigenvalue problems, one reason is in the difficulty in solving the EVPs. Polynomial eigenvalue problems are typically solved by linearization (6), (7).Rachůnková et al.,(8) study the solvability of large types of nonlinear singular problems for ordinary differential equations. Hammerling et al.,(9) concerned with the computation of eigenvalues and eigenfunctions of singular eigenvalue problems(EVPs) arising in ordinary differential equations, two different numerical methods to determine values for the eigenparameter such that the boundary value problem has nontrivial solutions are considered. The first approach incorporates a collocation method. The second solution approach represents a matrix method.Tawfiq and Mjthap[10]suggest a semi analytic technique to solve a class of singular eigenvalue problem. This paper suggest a semi-analytic technique to solve singular eigenvalue problems (SEVP′s) using osculator interpolation polynomial. That is, propose a series solution of singular eigenvalue problems with singularity of first-, second- and third- kinds by means of the osculator interpolation polynomial. The proposed method enables us to obtain the eigenvalue and corresponding nonzero eigenvector of the second order singular boundary value problem (SBVP).

2. Singular Boundary Value Problem

The general form of the second order two point boundary value problem (TPBVP) is:

$$y'' + P(x)\, y' + Q(x)\, y = 0, \quad a \le x \le b, \qquad (1)$$

$y(a) = A$ and $y(b) = B$, where $A, B \in R$

There are two types of a point $x_0 \in [0,1]$: Ordinary point and Singular point.

A function $y(x)$ is analytic at x_0 if it has a power series expansion at x_0 that converges to $y(x)$ on an open interval containing x_0. A point x_0 is an ordinary point of the ODE (1), if the functions $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise x_0 is a singular point of the ordinary differential equations. On the other hand if $P(x)$ or $Q(x)$ are not analytic at x_0 then x_0 is said to be a singular point [8].

Definition 1.1[11]

A TPBVP associated to the second order differential equation (2) is singular if one of the following situations occurs:

• 0 and/or 1 are infinite;

• fis unbounded at some $x_0 \in [0, 1]$;

• fis unbounded at some particular value of y or y′.

3. Other Singular Problems

Consider four kinds of singularities [12]:

- The first kind is the singularity at one of the ends of the interval [0, 1];
- The second kind is the singularity at both ends of the interval [0, 1];
- The third kind is the case of a singularity in the interior of the interval;
- The fourth and final kind is simply treating the case of a regular differential equation on an infinite interval.

In this paper, we focus of the first three kinds.

Note

One deals of this thesis is to solve singular eignvalue problems with specific boundary conditions as the form:

$$(x - a)^m\, y'' = \lambda\, f(x, y, y'); \; 0 \le x \le 1, \; a \in [0,1] \qquad (2)$$

where m is integer and f is nonlinear functions.

4. Osculator Interpolation Polynomial

In this paper we use two-point osculatory interpolation polynomial; essentially this is a generalization of interpolation using Taylor

polynomialP in which values of y and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P [13].

We are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0, 1], i.e., $P^{(j)}(x_i) = f^{(j)}(x_i)$, $j = 0, \ldots, n$, $x_i = 0, 1$, where a useful and succinct way of writing oscillatory interpolantP_{2n+1} of degree 2n + 1 was given for example by Phillips [14] as:

$$P_{2n+1}(x)=\sum_{j=0}^{n}\{y^{(j)}(0)q_j(x)+(-1)^j y^{(j)}(1)q_j(1-x)\},(3)$$

$$q_j(x)=(x^j/j!)(1-x)^{n+1}\sum_{s=0}^{n-j}\binom{n+s}{s}x^s=Q_j(x)/j!, (4)$$

so that (3) with (4) satisfies:

$$y^{(j)}(0)=P^{(j)}_{2n+1}(0),\ y^{(j)}(1)=P^{(j)}_{2n+1}(1),\ j=0,1,2,\ldots,n$$

.implying that P_{2n+1} agrees with the appropriately truncated Taylor series for *y* about *x* = 0 and *x* = 1. We observe that (3) can be written directly in terms of the Taylor coefficients a_i and b_i about *x* = 0 and *x* = 1 respectively, as:

$$P_{2n+1}(x) = \sum_{j=0}^{n}\{a_j Q_j(x) + (-1)^j b_j Q_j(1-x)\},(5)$$

5. Solution of Second Order Singular Eigenvalue Problems

In this section, we suggest a semi-analytic techniquewhich is based on oscillatory interpolating polynomials P_{2n+1}andTaylor series expansion to solve 2nd order singular eigenvalue Problems (SEVP′s).

polynomials. The idea is to approximate a function *y* by a

A general form of 2nd order SEVP's is (if the singular point is x = 0):

$$x^m y''(x) = \lambda f(x, y, y'), 0 \le x \le 1; \quad (6a)$$

where f are in general nonlinear functions of their arguments and m is integer.

Subject to the boundary condition(BC):

In the case Dirichlet BC: y(0)= A,y(1) = B,where A, B ϵ R (6b)

In the case Neumann BC:y′(0)= A,y′(1) = B,whereA, Bϵ R (6c)

In the case Cauchy or mixed BC: y(0)= A, y′(1) = B, where A, B ϵ R (6d).Or

y′(0)= A, y(1) = B, where A, B ϵ R (6e)

Now, to solve this problems by suggested method doing the following steps:

Step one:-

Evaluate Taylor series of y(x) about x = 0, i.e.,

$$y=\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad (7a)$$

where $y(0) = a_0$, $y'(0) = a_1$, $y''(0) / 2! = a_2$, …, $y^{(i)}(0) / i! = a_i$, i= 3, 4,…

And evaluate Taylor series of y(x) about x = 1, i.e.,

$$y = \sum_{i=0}^{\infty} b_i (x-1)^i = b_0 + b_1(x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \ (7b)$$

where $y(1) = b_0$, $y'(1) = b_1$, $y''(1) / 2! = b_2$, … , $y^{(i)}(1) / i! = b_i$, i = 3, 4,…

Step two:-

Insert the series form (7a) into equation (6a) and put x = 0, then equate the coefficients of powers of x to obtain a_2.

Insert the series form (7b) into equation (6a) and put x = 1, then equate the coefficients of powers of (x-1) to obtain b_2.

Step three:-

Derive equation (6a) with respect to x, to get new form of equation say (8) as follows:

$$m x^{m-1} y''(x) + x^m y'''(x) = \lambda \frac{df(x,y,y')}{dx} \quad ,(8)$$

then, insert the series form (7a) into equation (8) and put x = 0 and equate the coefficients of powers of x to obtain a_3, again insert the series form (7b) into equation (8) and put x = 1, then equate the coefficients of powers of (x-1) to obtain b_3.

Step four:-

Iterate the aboveprocess many times to obtain a_4, b_4 then a_5, b_5 and so on, that is, to get a_i and b_i for all $i \geq 2$, the resulting equations can be solved using MATLAB version 7.10, to obtain a_i andb_i for all $i \geq 2$.

Step five:-

The notation implies that the coefficients depend only on the indicated unknown's a_0, a_1, b_0, b_1, and λ, use the BC′s to get two coefficients from these, therefore, we have only two unknown coefficients and λ. Now, we can construct two point osculatory interpolating polynomial $P_{2n+1}(x)$ by insert these coefficients (a_iś and b_iś) as the following:

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ a_j \, Q_j(x) + (-1)^j \, b_j \, Q_j(1-x) \}, \ (9)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x)/j!,$$

Step six:-

To find the unknowns coefficients integrate equation (6a) on [0, x] to obtain:

$$x^m y'(x) - m\, x^{m-1} y(x) + m(m-1)\int_0^x s^{m-2}\, y(s)\, ds - \lambda \int_0^x f(s, y, y')\, ds = 0, \qquad (10a)$$

and again integrate equation (10a) on [0, x] to obtain:

$$x^m y(x) - 2m\int_0^x s^{m-1} y(s)\, ds + m(m-1)\int_0^x (1-s)s^{m-2} y(s) ds + \lambda\int_0^x (1-s) f(s, y, y') = 0, \quad (10b)$$

Step seven:-

Putting x = 1 in Equation (10) to get:

$$b_1 - m b_0 + m(m-1)\int_0^1 s^{m-2}\, y(s)\, ds + \lambda \int_0^1 f(s, y, y')\, ds = 0, \qquad (11a)$$

and

$$b_0 - 2m\int_0^1 s^{m-1} y(s)\, ds + m(m-1)\int_0^1 (1-s)s^{m-2}\, y(s)\, ds + \lambda \int_0^1 (1-s) f(s, y, y') ds = 0, (11b)$$

Step eight:-

Use $P_{2n+1}(x)$ which constructed in step five as a replacement of y(x), we see that Equation (11) have only two unknown coefficients from a_0, a_1, b_0, b_1 and λ. If the BC is Dirichlet boundarycondition, that is, we have a_0 and b_0, then Equation (11) have two unknown coefficients a_1, b_1 and λ. If the BC is Neumann, that is, we have a_1 and b_1, then equations (11) have two unknown coefficients a_0, b_0 and λ.

Finally, if the BC is Cauchyor mixed condition, i.e., we have a_0 and b_1 or a_1 and b_0, then

Stepnine:-

In the case Dirichlet BC, we have:

Equation (11) have two unknown coefficients a_1, b_0 or a_0, b_1 and λ.

$$F(a_1, b_1, \lambda) = b_1 - mb_0 + m(m-1)\int_0^1 s^{m-2}\, y(s)\, ds + \lambda \int_0^1 f(s, y, y')\, ds = 0, \qquad (12a)$$

$$G(a_1, b_1,\lambda) = b_0 - 2m\int_0^1 s^{m-1} y(s)ds + m(m-1)\int_0^1 (1-s)\, s^m s^{m-2} y(s)ds + \lambda\int_0^1 (1-s)f(s, y, y')ds = 0, (15b)$$

$$^2 y(s)ds + \lambda\int_0^1 (1-s)f(s, y, y')ds = 0, (12b)$$

$$(\partial F/\partial a_1)(\partial G/\partial b_1) - (\partial F/\partial b_1)(\partial G/\partial a_1) = 0, \ (12c)$$

In the case Neumann BC, we have:

$$F(a_0, b_0, \lambda) = b_1 - mb_0 + m(m-1)\int_0^1 s^{m-2}\, y(s)\, ds + \lambda\int_0^1 f(s, y, y')\, ds = 0, \quad (13a)$$

$$G(a_0, b_0, \lambda) = b_0 - 2m\int_0^1 s^{m-1} y(s)ds + m(m-1)\int_0^1 (1-s)\, s^{m-2} y(s)ds + \lambda\int_0^1 (1-s)f(s, y, y')ds = 0 (13b)$$

$$(\partial F/\partial a_0)(\partial G/\partial b_0) - (\partial F/\partial b_0)(\partial G/\partial a_0) = 0, \ (13c)$$

In the case mixed BC, we have:

$$F(a_1, b_0, \lambda) = b_1 - mb_0 + m(m-1)\int_0^1 s^{m-2}\, y(s)\, ds + \lambda\int_0^1 f(s, y, y')\, ds = 0, \quad (14a)$$

$$G(a_1, b_0, \lambda) = b_0 - 2m\int_0^1 s^{m-1} y(s)ds + m(m-1)\int_0^1 (1-s)\, s^{m-2} y(s)ds + \lambda\int_0^1 (1-s)f(s, y, y')ds = 0 \ (14b)$$

$$(\partial F/\partial a_1)(\partial G/\partial b_0) - (\partial F/\partial b_0)(\partial G/\partial a_1) = 0, \quad (14c)$$

Or

$$F(a_0, b_1, \lambda) = b_1 - mb_0 + m(m-1)\int_0^1 s^{m-2}\, y(s)\, ds + \lambda\int_0^1 f(s, y, y')\, ds = 0, \quad (15a)$$

$$G(a_0, b_1, \lambda) = b_0 - 2m\int_0^1 s^{m-1} y(s)ds + m(m-1)\int_0^1 (1-s)$$

$$(\partial F/\partial a_0)(\partial G/\partial b_1) - (\partial F/\partial b_1)(\partial G/\partial a_0) = 0, \ (15c)$$

So, we can find these coefficients by solving the system of algebraic Equation (12) or (13) or (14) or (15) using MATLAB, so insert the value of the unknown coefficients into equation (9), thus equation (9) represent the solution of the problem.

6. Example

In this section, we investigate the method using example of singular eigenvalue problem. The algorithm was implemented in MATLAB 7.10.

The bvp4c solver of MATLAB has been modified accordingly so that it can solvesome class of singular eigenvalue problem as effectively as it previously solved eigenvalue problem.

Also, we report a more conventional measure of the error, namely the error relative to the larger of the magnitude of the solution component and taking advantage of having a continuous approximate solution, we report the largest error found at 10 equally spaced points in [0, 1].

The problem is an application of oxygen diffusion:

$$y'' + (1 + \frac{1}{x})y' (+\frac{5x^3(5x^5e^y - x - \lambda - 4)}{4 + x^5}) = 0,$$

with B.C (Neumann case): $y'(0) = 0$, $y'(1) = -1$, the exact solution is (Tawfiq, L. N. M. et al): $y = -\ln(x^5 + 4)$.

Now, we solve this problem by suggested method, we have the following unknowns coefficients a_0, b_0, a_1, b_1 and λ, we got a_1 and b_1 from BC′s, then from Equation (13), we have (using MATLAB): $a_0 = -1.386294361119891$, $b_0 = -1.6094379124341$ and $\lambda = 1$.

Then from Equation (9), we have (for n = 7):

$P_{15} = -0.1142318896x^{15} + 0.7852891723x^{14} - 2.23317271x^{13} + 3.413936838x^{12} - 3.087794733x^{11} + 1.697854523x^{10} - 0.4991720457x^{9} + 0.06414729306x^{8} - 0.25x^{5} - 1.386294361$.

For more details, Table (1) give the results for different nodes in the domain, for n = 7 and Figure (1) illustrate suggested method for n = 7.

Abukhaled et al.,[15] applying L'Hopital's rule to overcome the singularity at x = 0 and then the modified spline approach are used and got maximum error $7.79e^{-4}$ and resolution this problem using finite difference method then gave the maximum error $1.46e^{-3}$,but solving this problem by suggested method gave the maximum error $9.399395723974635e^{-007}$ see Table (1).The proposed method superiority isevident here.

7. Error / Defect Weights

Every known BVP software package reports an estimate of either the relative error or the maximum relative defect. The weights used to scale either the error or the maximum defect differs among BVP software. Therefore, the BVP component of pythODE allows users to select the weights they wish to use. The default weights depend on whether an estimate of the error or maximum defect is being used. If the error is being estimated, then the BVP component of pythODE uses [11]. In this paper, we modify this package to consist SEVP′s with named "pythSEVPODE", which defined as:

$$\frac{\|y(x) - p(x)\|_\infty}{1 + \|p(x)\|_\infty}; \qquad 0 \le x \le 1, \ (16)$$

where y(x) is exact solution and P(x) is suggested solution of SEVP′s.

If the maximum defect is being estimated, then the SEVP′s component of "pythSEVPODE" uses:

$$\frac{\left\|p''_{2n+1}(x) - \frac{\lambda}{x} f(x, p(x), p'(x))\right\|_\infty}{1 + \left\|\frac{\lambda}{x} f(x, p(x), p'(x))\right\|_\infty}; \qquad (17)$$

The relative estimate of both the error and the maximum defect are slightly modified from the one used in BVP SOLVER.

Now, apply package (17) for the above example as follows:

$$\frac{\left\|p''_{15} - \frac{\lambda}{x} f(x, p_{15}, p'_{15})\right\|_\infty}{1 + \left\|\frac{\lambda}{x} f(x, p_{15}, p'_{15})\right\|_\infty} = \frac{0.50000000000000097}{1+3} = 0.142857142857171$$

8. Conclusions

In the present paper, we have proposed a semi-analytic technique to solve second order singular eigenvalue problems. By using oscillator interpolation, the result shown that the Semi - Analytic technique can be used successfully for finding the solution of singular eigenvalue problem with boundary conditions of second order with singular point of first, second and third kind. It may be concluded that this technique is a very powerful and efficient in finding highly accurate solutions for a large class of differential equations.

Finally, The bvp4c solver of MATLAB has been modified accordingly so that it can solve some class of singular eigenvalue problem with boundary conditions as effectively as it previously solved non-singular BVP.

Table 1: The exact and suggested solution for n = 7 of Example

x_i	Exact solution y(x)	Suggested solutionP_{15}	Errors $\| y(x) - P_{15} \|$
0	-1.38629436111989	-1.3862943611198	$4.440892098500626e^{-016}$
0.1	-1.3862968611165	-1.3862968608354	$2.812807764485115e^{-010}$
0.2	-1.38637435792006	-1.3863743295773	$2.834240797611187e^{-008}$
0.3	-1.3869016766664	-1.3869014277067	$2.489597183963355e^{-007}$
0.4	-1.38885108990158	-1.3888503799276	$7.099738041915771e^{-007}$
0.5	-1.3940765015619	-1.3940755616223	$9.399395723974635e^{-007}$
0.6	-1.40554781804177	-1.4055471943550	$6.23686427614345e^{-007}$
0.7	-1.4274530989357	-1.4274529126759	$1.862599887658689e^{-007}$
0.8	-1.46503160165727	-1.4650315848645	$1.679306937951708e^{-008}$
0.9	-1.5239867721873	-1.5239867720735	$1.136124527789661e^{-010}$
1	-1.6094379124341	-1.6094379124340	$2.220446049250313e^{-016}$
S.S.E= 1.874293078482109e-012 Max. error= 9.399395723974635e-007			

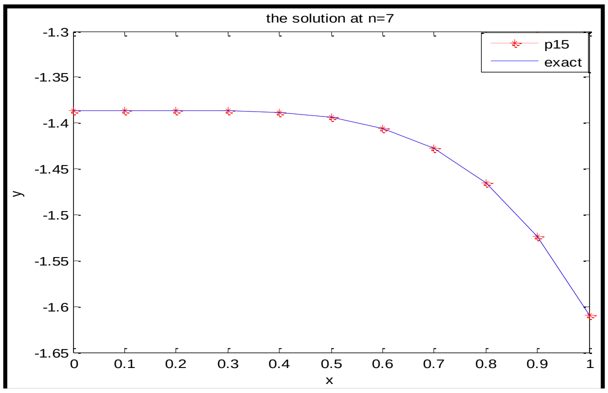

Figure 1: Comparison between the exact and suggestedsolutionof example

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كفاءة التقنية شبه – التحليلية في حل مسائل القيم الذاتية المنفردة ذات الشروط الحدودية

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الخلاصة

في هذا البحث نقترح التقنية شبه – التحليلية لحل مسائل القيم الذاتية المنفردة و من الرتبة الثانية ذات شروط حدودية باستخدام الاندراج التماسي التقنية تجد القيم الذاتية و المتجهات الذاتية الغير الصفرية المقابلة لها و التي تمثل الحل للمعادلة ضمن مجالها. تم عرض مثال توضيحي يعزز و يوضح التوقعات النظرية و كذلك المقارنة بين التقنية المقترحة و طرق أخرى.