Fuzzy KUS-ideals of KUS-algebras

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Abstract

The aim of this paper is introduce the notion of fuzzy KUS-ideal of KUS-algebra , several theorems , properties are stated and proved. The fuzzy relations on KUS-algebras are also studied.

Keywords: KUS-algebra, fuzzyKUS- sub-algebra, fuzzy KUS-ideal,homomorphisms of KUS-algebras, image and pre-image of fuzzyKUS-ideals, the Cartesian product of fuzzy KUS-ideals.

1. Introduction

A BCK-algebra is an important class of logical algebras introduced by [(1)]andwas extensively investigated by several researchers. The class of all BCK-algebras is aquasivariety. K. Is´eki posed an interesting problem whetherthe class of BCK-algebras is a variety. In[(6,7)] introduce the notion of KU-algebras.They gave the concept of homomorphism of KU-algebras and investigated some related properties .

The concept of fuzzy subset and various operations on it were first introducedby [(2)] , then fuzzy subsets have been applied to diverse field.The study of fuzzy subsets and their application to mathematical contexts hasreached to what is now commonly called fuzzy mathematics. Fuzzy algebra is animportant branch of fuzzy mathematics. The study of fuzzy algebraicstructureswas started with theintroduction of the concept of fuzzy subgroups in 1971 by [(3)]. Since these ideas have been applied to other algebraicstructures such as semi-groups, rings,ideals, modules and vector spaces. O. G. Xi,et al (1985-1991)in [(4)] applied this concept to BCK-algebra , and he introduced thenotion of fuzzy sub-algebras (ideals) of the BCK-algebras with respect to minimum , and since then Y. B. Jun, et al [(5)], studied fuzzy ideals and moreover several fuzzy structures in BCK-algebras are considered. SamyM. Mostafa, et al in [(6)]have introduced the notion of KUS-algebras,KUS-ideals, KUS-sub-algebrasand investigates the relations among them.In this paper , we introduce the notion of fuzzy KUS-ideals in KUS-algebras and then we investigate several basic properties which are related to fuzzy KUS-ideals. We describe how to deal with the homomorphism of image and inverse image of fuzzy KUS-ideals. we have also proved that the cartesian product of fuzzy KUS-ideals is a fuzzy KUS-ideal .

2. Preliminaries

Now we give some definitions and preliminary results needed in the later sections.

Definition 2.1[(6)]. Let $(X;*,0)$ be an algebra with a singlebinary operation ($*$). X is called a KUS-algebra if it satisfies the following identities: for any x, y, z $\in$X ,

(kus_1) :$(z*y)*(z*x) = y*x$,

(kus_2) : $0*x = x$,

(kus_3) : $x*x = 0$,

(kus_4) : $x*(y*z) = y*(x*z)$.

In X we can define a binary relation ($\leq$) by: $x \leq y$ if and only if $y*x = 0$.

In what follows, let $(X;*,0)$ denote aKUS-algebra unless otherwise specified.For brevity we also call X a KUS-algebra.

Lemma 2.2[(6)]. In any KUS-algebra$(X;*,0)$, the following properties hold: for all x, y, z $\in$X;

a) $x*y = 0$ and $y*x = 0$ imply $x = y$,

b) $y*[(y*z)*z]= 0$,

c) $(0*x)*(y*x) = y*0$,

d) $x = 0*(0*x)$,

e) $(x*y)*0 = y*x$,

f) $x \leq y$ implies that $y*z \leq x*z$,

g) $x \leq y$ implies that $z*x \leq z*y$,

h) $x \leq y$ and $y \leq z$ imply $x\leq z$,

i) $x*y \leq z$implies that $z*y\leq x$.

Example 2.3.

1) Let X = {0, 1, 2, 3} in which ($*$) is defined by the following table :

$*$	0	1	2	2
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

It is easy to show that $(X;*,0)$ is a KUS-algebra .

2) Let X = {0, 1, 2, 3, 4} in which ($*$) is defined by the following table :

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

It is easy to show that $(X;*,0)$ is a KUS-algebra .

Definition 2.4[(6)]. Let X be a KUS-algebra and let S be a nonempty subset of X.S is called a KUS-sub-algebra of X if $x*y \in S$ whenever $x \in S$ and $y \in S$.

Definition 2.5[(6)]. A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for x , y, z $\in$ X,

($Ikus_1$) ($0 \in I$) ,

($Ikus_2$) $(z*y)\in I$ and $(y*x)\in I$ imply

$(z*x) \in I$.

Example 2.6 . Let X ={0 ,a, b, c} in which ($*$) is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X;*,0)$ is a KUS-algebra . It is easy to show that I_1 ={0,a},I_2 ={0,b}, I_3={0,c}, and I_4 ={0, a, b,c} are KUS-ideals of X .

Proposition 2.7[6]. Every KUS-ideal of KUS-algebra X is a KUS-sub-algebra.

Proposition 2.8[6]. Let $\{I_i | i \in \Lambda\}$ be a family of KUS-ideals of KUS-algebra X. The intersection of any set of KUS-ideals of KUS-algebra X is also aKUS-ideal .

Definition 2.9[7,8]. Let $(X ;*,0)$ and $(Y; *`,0`)$ benonempty sets. The mapping $f : (X;*,0) \rightarrow (Y;*`,0`)$is called a homomorphismif itsatisfies:
$f(x*y) = f(x)*` f(y)$, for all x, y $\in$X.The set $\{x \in X| f(x) = 0'\}$is called the Kernel of f denoted by Ker f .

Theorem 2.10[6]. Let

$f : (X;*,0) \rightarrow (Y;*`,0`)$ be a homomorphism of a KUS-algebra X into a KUS-algebra Y, then :

A. $f(0) = 0'$.

B. f is injective if and only if Ker f ={0}.

C. $x \leq y$ implies $f(x) \leq f(y)$.

Theorem 2.11[5]. Let
$f : (X;*,0) \rightarrow (Y;*`,0`)$ be a homomorphism of a KUS-algebra X into a KUS-algebra Y, then :

(F_1) If S is a KUS-sub-algebra of X, then f (S) is a KUS-sub-algebra of Y.

(F_2) If I is a KUS-ideal of X, then f (I) is a KUS-ideal in Y.

(F_3) If Bis a KUS-sub-algebra of Y, then f^{-1}(B)is a KUS-sub-algebra of X.

(F_4) If J is a KUS- ideal in Y, then f^{-1}(J) is a KUS-ideal in X .

(F_5) Ker f is KUS-ideal of X.

(F_6) Im(f) is a KUS-sub-algebra of Y.

3. Fuzzy KUS-ideals and Homomorphismof KUS-algebras

In this section , we will discuss a new notion called fuzzy KUS-ideals of KUS-algebras and study several basic properties which are related to fuzzyKUS-ideals .

Definition 3.1[2]. Let $(X;*,0)$ X be a nonempty set, a fuzzy subset μin X is a function $\mu : X \rightarrow [0,1]$.

Definition 3.2.Let $(X;*,0)$ be a KUS-algebra , a fuzzy subset μ in X is called a fuzzyKUS-sub-algebra of X if for all x , y $\in$X,$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 3.3.Let $(X;*,0)$ be a KUS-algebra , a fuzzy subset μ in X is called a fuzzyKUS-ideal of X if it satisfies the following conditions: , for all x , y, z $\in$X ,

($Fkus_1$) $\mu(0) \geq \mu(x)$,

($Fkus_2$) $\mu(z*x) \geq \min\{\mu(z*y), \mu(y*x)\}$.

Example 3.4.

1) Let X ={0, 1, 2, 3} in which ($*$) is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X ; $*$, 0) is KUS-algebra . Define a fuzzy subset $\mu : X \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.7 & if\ x \in \{0,1\} \\ 0.3 & otherwise \end{cases}$$

I_1 = {0, 1} is aKUS-ideal of X.Routine calculation gives that μ is a fuzzy KUS-ideal of KUS-algebras X.

2) Consider X = {0, a, b, c,d} with($*$)defined by the table :

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X ;$*$,0) is a KUS-algebra. Define a fuzzy subsetμ:X → [0,1]such that μ(0) = t_1 , μ(a) = μ(b) = μ(c) = μ(d) = t_2 ,where $t_1, t_2 \in [0, 1]$ and $t_1 > t_2$.

Routine calculation gives that μ is a fuzzy KUS-ideal of KUS- algebra X .

Note that : To check that the axioms are satisfied on the given examples we use the program fisted in the index .

Lemma 3.5. Let μ be a fuzzy KUS-ideal in KUS-algebra X and if$x \le y$, then $\mu(x) \ge \mu(y)$, for all x, y∈ X.

Proof: Assume that$x \le y$, then$y * x=0$, and $\mu(0 * x) = \mu(x)$

$\ge \min \{\mu(0 * y), \mu(y * x)\}$

$=\min \{\mu(y), \mu(0)\}= \mu(y)$.

Hence $\mu(x) \ge \mu(y)$.⌂

Proposition 3.6. Let μ be a fuzzy KUS-ideal of KUS-algebra X, if the inequality$y * x \le z$ holds in X , then $\mu(x) \ge \min \{\mu(y), \mu(z)\}$, for all x, y, z ∈ X.

Proof: Assume that the inequality $y*x \le z$ holds in X, then by lemma(3.5), $\mu(y * x) \ge \mu(z)$--- (1).

By($Fkus_2$) ,$\mu(z * x) \ge \min \{\mu(z * y), \mu(y * x)\}$.Put z=0 ,then$\mu(0 * x) = \mu(x) \ge \min\{\mu(0 * y), \mu(y * x)\}$

$= \min\{\mu(y), \mu(y * x)\}$. Hence

$\mu(x) \ge \min \{\mu(y), \mu(y * x)\}$--- (2).

From (1) and (2), we get $\mu(x) \ge \min \{\mu(y), \mu(z)\}$, for all x, y, z ∈X. ⌂

Definition 3.7[(2)]. Let X be a nonempty set andμ be a fuzzy subset in X,fort ∈[0,1] , the set $\mu_t = \{ x \in X |\ \mu(x) \ge t\}$is called a level subset of μ .

Theorem3.8. Let μ be a fuzzy KUS-ideal in KUS-algebra X. μ is a fuzzyKUS-ideal of Xif and only if , for every t ∈[0,1] , μ_t is either empty or aKUS-ideal of X .

Proof: Assume that μ is a fuzzy KUS-ideal of X , by ($Fkus_1$) , we have$\mu(0) \ge \mu(x)$ for all x ∈X therefore $\mu(0) \ge \mu(x) \ge t$ for $x \in \mu_t$ and so $0 \in \mu_t$. Let x, y, z ∈ X be such that $(z * y) \in \mu_t$ and $(y * x) \in \mu_t$, then$\mu(z * y) \ge t$ and $\mu(y * x) \ge t$, since μ is a fuzzy KUS-ideal, it follows that $\mu(z * x) \ge \min \{\mu(z * y), \mu(y * x)\} \ge t$ andwe have that$x * z \in \mu_t$. Henceμ_t is aKUS-ideal of X .

Conversely , we only need to show that ($Fkus_1$) and ($Fkus_2$) are true . If($Fkus_1$) is false , then there exist x`∈X such that μ (0) < μ(x`). If wetake t`=(μ (x`) + μ (0))/2, then μ(0) <t` and 0 ≤ t` < μ (x`) ≤ 1 ,thenx`∈μ and μ ≠ ϕ. As $\mu_{t`}$ is aKUS-ideal of X , we have 0 ∈ $\mu_{t`}$ and so μ (0) ≥ t`. This is a contradiction . Now , assume ($Fkus_2$) is nottrue ,then there exist x` , y` , z` ∈ X such that ,
μ (z`∗x`) <min{μ(z`∗y`),μ (y`∗x`)}.
Puttingt`=(μ(z`∗x`)+min{μ(z`∗y`), μ (y` ∗ x`)}/2 , thenμ (x` ∗z`) < t`and
0 ≤ t` < min {μ (z`∗x`), μ(y`∗x`)} ≤ 1, henceμ (z`∗y`))> t` and μ(y`*x`) > t`,which imply that
(z`∗y`) ∈ $\mu_{t`}$ and(y`∗x`) ∈ $\mu_{t`}$ .Since $\mu_{t`}$ is aKUS-ideal ,it follows that(x`∗ z`)∈ $\mu_{t`}$ and thatμ (x`∗z`)≥t`, thisis also a contradiction . Hence μ is a fuzzy KUS-ideal of X .⌂

Corollary3.9. Letμ be a fuzzy subset inKUS-algebra X. If μis a fuzzyKUS-ideal, thenfor every t ∈Im (μ) , μ_t is a KUS- ideal of X when $\mu_t \neq \varnothing$.

Theorem 3.10. Letμ be a fuzzy subset in KUS-algebra X . If μis a fuzzy KUS-sub-algebra of X if and only if , for every t ∈[0,1] , μ_t is either empty or a KUS-sub-algebra of X .

Proof: Assume that μ is a fuzzy KUS-sub-algebra of X ,let x, y ∈ X be such that x∈ μ_t andy∈ μ_t , then μ (x) ≥ t and μ (y) ≥ t .Since μ is a fuzzy KUS-sub-algebra, it follows that μ (x∗y) ≥ min {μ (x) , μ (y)} ≥ t and that (x∗y)∈ μ_t . Hence μ_t is a KUS-sub-algebra of X .

Conversely, assumeμ (x∗y) ≥ min {μ(x), μ(y)}is not true ,then there exist x`and y`∈X such that ,μ (x`∗y`) < min {μ (x`) , μ (y`)}.
Putting t`=(μ (x`∗ y`) + min {μ(x`), μ (y` )}/2 , then μ (x`) < t` and 0 ≤ t` < min{μ (x`), μ (y`)}≤1, hence μ (x`) > t` and μ(y`)> t`, which imply that x`∈ $\mu_{t`}$ andy` ∈ $\mu_{t`}$, since $\mu_{t`}$ is a KUS-sub-algebra ,it follows that x`∗y`∈ $\mu_{t`}$ and that μ (x` ∗y`) ≥ t`, this is also a contradiction . Hence μ is a fuzzy KUS-sub-algebra of X . ⌂

Proposition 3.11. Every fuzzy KUS-ideal of KUS-algebra X is a fuzzy KUS-sub-algebraof X**.**

Proof: Since μ is fuzzy KUS-ideal of a KUS-algebra X , then by proposition (3.8), for every t ∈[0,1] , μ_t is either empty or a KUS-ideal of X. By proposition(2.7), for every t ∈[0,1], μ_t is either empty or a KUS-sub-algebra of X .Henceμ is a fuzzy KUS-sub-algebra of KUS-algebra X by proposition (3.10) . ⌂

Theorem 3.12. Let A be an KUS-ideal of KUS-algebra X.Then for any fixed number t in an open interval (0,1) , there exists a fuzzy KUS- ideal μ of X such that μ_t = A .

Proof: Define μ: X →[0,1] by

$$\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} .$$

Where t is a fixed number in (0, 1). Clearly, μ(0) ≥μ(x) for all x ∈X .Let x,

y, z$\in$X. If $(y*x) \notin A$, then $\mu(y*x)=0$ and so $\mu(z*x) \geq 0 = \min\{\mu(z*y),\mu(y*x)\}$. If $(z*y)\in A$, thenclearly $\mu(z*x) \geq \min\{\mu(z*y),\mu(y*x)\}$.
If $(z*x) \notin A$, $(y*x) \in A$, then $(z*y)\notin A$, since A is a KUS-ideal.Thus$\mu(z*x)=0=\min\{\mu(z*y),\mu(y*x)\}$.
Hence$\mu$is a fuzzy KUS-ideal of X. It is clear that $\mu_t = A$.⌂

Theorem 3.13. Let A be a nonempty subset of a KUS-algebra X and μ be a fuzzy subset of X such that μ is into $\{0, 1\}$, so that μ is the characteristic function of A . Then μ is a fuzzy KUS-ideal in X if and only if A is a KUS-ideal of X .

Proof: Assume that μ is a fuzzy KUS-ideal in X, since $\mu(0) \geq \mu(x)$ for all $x \in X$,clearly we have $\mu(0)=1$, and so $0 \in A$. Let x, y, z $\in$X be such that$(z*y)\in A$ and $(y*x)\in A$, since μ is a fuzzy KUS-ideal of X , it follows that$\mu(z*x)\geq \min\{\mu(z*y), \mu(y*x)\}=1$, and $\mu(z*x)=1$.
This mean that $(z*x)\in A$, i.e., A is aKUS-ideal of X .
Conversely, suppose A is a KUS-ideal of X, since $0 \in A$, $\mu(0)= 1 \geq\mu(x)$, for all $x\in X$. Let x, y, z $\in$ X,
If$(y*x)\notin A$, then $\mu(y*x) = 0$, and so $\mu(z*x) \geq 0 = \min\{\mu(z*y), \mu(y*x)\}$.
If $(z*x)\notin A$, and $(y*x)\in A$, then$(z*y)\notin A$ (since A is KUS-ideal).
Thus $\mu(z*x) = 0 = \min\{\mu(z*y), \mu(y*x)\}$. Hence μ is a fuzzy KUS-ideal of X.⌂

Theorem 3.14. Let μbe a fuzzy KUS-ideal of a KUS-algebra X and letμ_{t_1}, μ_{t_2} be level KUS-ideals ofμ, where $t_1< t_2$, then the following are equivalent.
E_1) $\mu_{t_1} = \mu_{t_2}$.
E_2)There is no $x\in X$ such that$t_1 \leq \mu(x) <t_2$.

Proof: Assume that$\mu_{t_1} = \mu_{t_2}$, for $t_1< t_2$and that there exist $x \in X$ such that $t_1 \leq \mu(x) <t_2$. Then μ_{t_2} is proper subset of μ_{t_1}, a contradiction.
Conversely suppose that there is no $x\in X$ such that $t_1 \leq \mu(x) <t_2$. It follows from$t_1< t_2$,that$\mu_{t_1} \subset \mu_{t_2}$. Let$x \in \mu_{t_1}$. Then $\mu(x) \geq t_1$, and hence $\mu(x) \geq t_2$, because$\mu(x)$ does not lie between$t_1$and t_2. Hence$x \in \mu_{t_2}$, this implies that$\mu_{t_1} \subset \mu_{t_2}$. Therefore, $\mu_{t_1} = \mu_{t_2}$. ⌂

Proposition 3.15. The intersection of any set of fuzzy KUS-ideals of KUS-algebraX is also fuzzy KUS-ideal .

Proof: Let $\{\mu_i \mid i \in\Lambda.\}$ be a family of fuzzy KUS-ideals of KUS- algebra X , then for anyx, y, z $\in$X , i $\in\Lambda$,
$(\bigcap_{i\in\Lambda}\mu_i)(0) = \inf(\mu_i(0)) \geq \inf(\mu_i(x))$
$= (\bigcap_{i\in\Lambda}\mu_i)(x)$ and
$(\bigcap_{i\in\Lambda}\mu_i)(z*x) = \inf(\mu_i(z*x)) \geq \inf(\min\{\mu_i(z*y)), \mu_i(y*x)\})$
$= \min\{\inf(\mu_i(z*y)), \inf(\mu_i(y*x)\}$
$= \min\{(\bigcap_{i\in\Lambda}\mu_i)(z*y)), (\bigcap_{i\in\Lambda}\mu_i)(y*x)\}$.
This completes the proof . ⌂

Definition 3. 16(7,8). Let
$f : (X;*,0) \rightarrow(Y;*',0')$ be a mapping nonempty sets X andY respectively. If

μis a fuzzy subset of X, then the fuzzy subset β of Y defined by:

$$f(\mu)(y)=\begin{cases}\sup\{\mu(x): x\in f^{-1}(y)\} & if\ f^{-1}(y)=\{x\in X, f(x)=y\}\neq\phi\\ 0 \quad otherwise\end{cases}$$

is said to be the image of μunder f **.** Similarly if β is a fuzzy subset of Y , then the fuzzy subset μ = (β o f) in X (i.e thefuzzy subset defined by μ (x) = β (f (x)) for all x ∈X) is called the pre-image of βunder f .

Theorem 3.17.An into homomorphic pre-image of a fuzzy KUS-ideal is also afuzzy KUS-ideal .

Proof: Let f : (X;∗,0) →(Y;∗',0') be an into homomorphism of KUS-algebras, β a fuzzyKUS-ideal of Y and μ the pre-image of β under f , then β (f (x)) = μ (x) , for allx ∈X . Since f (x) ∈ Y and β is a fuzzy KUS-ideal of Y, it follows thatβ(0') ≥ β(f (x)) = μ(x) , for every x ∈X, where 0' is the zero element of Y.But β(0') = β(f (0)) = μ(0) and so μ(0) ≥ μ(x) for x ∈ X.

Now let x, y, z ∈ X , thenwe get

μ (z∗x)= β(f (z∗x))= β(f (z)∗` $f$ (x)) ≥min{β( $f$ (z)∗` f (y)),β(f (y)∗` f (x))} = min {β (f (z ∗y)),β(f (y∗ x))} = min{μ(z∗y)), μ (y∗x)}i.e., μ (z∗x) ≥min{μ(z∗y)), μ (y∗x)},for all x, y, z ∈ X. ⌂

Definition 3.18[7,8]**.** A fuzzy subset μ of a set X has sup propertyif for any subset T ofX , there exist t_0∈T such thatμ(t_0) = sup {μ(t)| t∈T}.

Theorem 3.19. Let f : (X;∗,0) →(Y;∗',0') be a homomorphism betweenKUS-algebras X and Y respectively. For every fuzzy KUS-ideal μ in Xwith sup property, f (μ) is a fuzzy KUS-ideal of Y.

Proof: By definition β(y') = f (μ)(y') := sup{μ(x)| $x\in f^{-1}(y^{\backslash})$ },for ally'∈ Y(sup∅ = 0).We have to prove that β(z'∗'x') ≥ min {β(z'∗'y'),β(y'∗'x')}, for all x', y', z' ∈ Y.

(I) Let f : (X;∗,0) →(Y;∗',0') be a onto homomorphism ofKUS-algebras,μ is a fuzzy KUS-idealof X with sup property and β the image of μ under f . Since μ is a fuzzy KUS-ideal of X , we have μ(0) ≥ μ(x) for all x ∈ X. Note that 0 ∈ f^{-1}(0'), where 0 and 0' are the zero elements of X and Y respectively.Thus

$$\beta(0^{\backslash})=\sup_{t\in f^{-1}(x^{\backslash})}\mu(t)=\mu(0)\geq\mu(x)$$

= μ(0) ≥ μ(x) for all x ∈ X, which impliesthat $\beta(0^{\backslash})\geq\sup_{t\in f^{-1}(x^{\backslash})}\mu(t)=\beta(x^{\backslash})$ for any $x^{\backslash}\in Y$. For any x', y', z' ∈ Y , let $x_0\in f^{-1}(x^{\backslash})$, $y_0\in f^{-1}(y^{\backslash})$, $z_0\in f^{-1}(z^{\backslash})$ be such that:

$$\mu(z_0*y_0)=\beta[f(z_0*y_0)]$$
$$=\beta[f(z^{\backslash}*y^{\backslash})]=\sup_{z_0*y_0\in f^{-1}(z^{\backslash}*y^{\backslash})}\mu(z_0*y_0)\quad and$$
$$\mu(y_0*x_0)=\beta[f(y_0*x_0)]$$
$$=\beta[f(y^{\backslash}*x^{\backslash})]=\sup_{y_0*x_0\in f^{-1}(y^{\backslash}*x^{\backslash})}\mu(y_0*x_0)$$

, then

$$\beta(z^{\backslash} * x^{\backslash}) = \sup_{t\in f^{-1}(z^{\backslash}*x^{\backslash})} \mu(t)$$
$$= \mu(z_0 * x_0)$$
$$\geq \min\{\mu(z_0 * y_0), \mu(y_0 * x_0)\}$$
$$= \min\left\{ \sup_{t\in f^{-1}(z^{\backslash}*y^{\backslash})} \mu(t) \quad , \quad \sup_{t\in f^{-1}(y^{\backslash}*x^{\backslash})} \mu(t) \right\}$$
$$= \min\{\beta(z^{\backslash} * y^{\backslash}), \beta(y^{\backslash} * x^{\backslash})\}$$

Hence β is a fuzzy KUS-ideal of Y.

(II) If f is not onto: For every $x^{\backslash} \in Y$, we define $X_{x^{\backslash}} := f^{-1}(x^{\backslash})$.Since f is a homomorphism,we get

$X_{z^{\backslash}} * X_{y^{\backslash}} \subset X_{z^{\backslash}*y^{\backslash}}$ *and* $X_{y^{\backslash}} * X_{x^{\backslash}} \subset X_{y^{\backslash}*x^{\backslash}}$,for all x', y', z' ∈ Y------------- (*).

Let x' ,y' ,z' ∈ Y be arbitrarily given. If $(z^{\backslash} * y^{\backslash}) \notin \mathrm{Im}(f) = f(X)$, then by definition $\beta(z^{\backslash} * y^{\backslash}) =$ 0. But if $(z^{\backslash} * y^{\backslash}) \notin f(X)$, i.e. , $X_{z^{\backslash}*y^{\backslash}} = \phi$, then by (*)at least one of $z^{\backslash}, y^{\backslash}, x^{\backslash} \notin f(X)$ and hence β(z'*'x') ≥ 0 = min {β(z'*'y'),β(y'*'x')}. ⌂

4. Cartesian product of fuzzy KUS-ideal

In this section , we will discuss , investigate a new notion calledCartesian product of fuzzy KUS-ideals and we study several basic properties which related to fuzzy KUS-ideals .

Definition 4.1 [(9)]. A fuzzy relation R on any set S is a fuzzy subset R: S×S→[0,1].

Definition 4.2 [(9)]. If R is a fuzzy relation on set S and β is a fuzzy subset of S , then R is a fuzzy relation on β if R(x,y) ≤ min {β(x), β(y)}, for all x, y∈S .

Definition 4.3 [(9)]. Let μ and β be fuzzy subsets of a set S . The Cartesian product of μ and β is defined by:($\mu\times\beta$)(x,y)=min{μ(x), β(y)}, for all x, y∈S .

Lemma 4.4 [(9)]. Let μ and β be fuzzy subsets of a set S . Then,

(1) ($\mu\times\beta$) is a fuzzy relation on S ,

(2) $(\mu\times\beta)_t = \mu_t \times \beta_t$, for all t ∈[0,1].

Definition 4.5 [(9)]. If β is a fuzzy subset of a set S , the strongest fuzzy relation on S,that is, a fuzzy relation on β is R_β given by R_β (x,y) = min {β(x) , β(y)}, for all x, y∈S .

Lemma 4.6 [(9)]. For a given fuzzy subset β of a set S, let R_β be the strongest fuzzy relation on S. Then for t ∈[0,1], we have $(R_\beta)_t = \beta_t \times \beta_t$.

Proposition 4.7. For a given fuzzy subset β of a KUS-algebra X , let R_β be the strongest fuzzy relation on X . If β is a fuzzy KUS-ideal of $X\times X$, then R_β (x,x) ≤ R_β (0,0), for all x∈X .

Proof: Since R_β is a strongest fuzzy relation of $X\times X$, it follows from that, $R_\beta(x,x)$ = min {β(x),β(x)}≤min{β(0), β(0)}= $R_\beta(0,0)$, which implies that R_β (x,x) ≤ R_β(0,0) .⌂

Proposition4.8. For a given fuzzy subset β of a KUS-algebra X , let R_β be the strongest fuzzy relation on X . If

R_β is a fuzzy KUS-ideal of $X\times X$, then $\beta(x) \le \beta(0)$, for all $x\in X$.

Proof: Since R_β is a fuzzy KUS-ideal of $X\times X$, it follows from ($Fkus_1$), R_β (x,x) $\le R_\beta$(0,0) . Where (0,0) is the zero element of X×X. But this means that, $\min\{\beta(x), \beta(x)\} \le \min\{\beta(0), \beta(0)\}$ which implies that $\beta(x) \le \beta(0)$. ⌂

Remark 4.9. Let X and Y be KUS-algebras, we define ($*$) on $X\times Y$ by: for all $(x,y),(u,v)\in X\times Y$,
$(x,y)*(u,v) = (x*u,y*v)$. Then clearly $(X\times Y,*,(0,0))$ is a KUS-algebra.

Theorem 4.10. Let μ and β be fuzzy KUS-ideals of KUS-algebra X. Then $\mu\times\beta$ is afuzzy KUS-ideal of $X\times X$.

Proof: Note first that for every $(x,y)\in X\times X$, $(\mu\times\beta)(0,0) = \min\{\mu(0), \beta(0)\} \ge \min\{\mu(x), \beta(y)\} = (\mu\times\beta)(x,y)$.

Now let(x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X\times X$. Then
$(\mu\times\beta)(z_1*x_1,z_2*x_2)= \min\{\mu(z_1*x_1), \beta(z_2*x_2)\} \ge \min\{\min\{\mu(z_1*y_1), \mu(y_1*x_1)\}, \min\{\beta(z_2*y_2), \beta(y_2*x_2)\}\}$
$= \min\{\min\{\mu(z_1*y_1), \beta(z_2*y_2)\}, \min\{\mu(y_1*x_1),\beta(y_2*x_2)\}\}$
$= \min\{(\mu\times\beta)(z_1*y_1, z_2*y_2), (\mu\times\beta)(y_1*x_1,y_2*x_2)\}$
Hence $(\mu\times\beta)$ is a fuzzy KUS-ideal of $X\times X$. This completes the proof. ⌂

Theorem 4.11. Let μ and β be fuzzy subsets of KUS-algebra X such that $\mu\times\beta$ is a fuzzy KUS-ideal of $X\times X$. Then , for all $x\in X$
(i) either $\mu(0) \ge \mu(x)$ or$\beta(0) \ge \beta(x)$.
(ii) $\mu(0) \ge \mu(x)$, for all $x\in X$,then either $\beta(0) \ge \mu(x)$ or$\beta(0) \ge \beta(x)$.
(iii) If $\beta(0) \ge \beta(x)$, for all $x\in X$, then either $\mu(0)\ge \mu(x)$or $\mu(0) \ge \beta(x)$.
(iv) Either μ or β is a fuzzy KUS-ideal of X .

Proof:
(i) suppose that$\mu(x)>\mu(0)$ and $\beta(y)>\beta(0)$,for some$x , y \in X$. Then $(\mu\times\beta)(x,y) = \min\{\mu(x), \beta(y)\}>\min\{\mu(0), \beta(0)\} = (\mu\times\beta)(0,0)$.This is a contradiction and we obtain (i).
(ii) Assume that there exist $x , y \in X$ such that $\mu(x) > \beta(0)$ and $\beta(y) > \beta(0)$. Then
$(\mu\times\beta)(0,0) = \min\{\beta(0),\beta(0)\} = \beta(0)$ it follows that
$(\mu\times\beta)(x,y) = \min\{\mu(x), \beta(y)\}> \beta(0)= (\mu\times\beta)(0,0)$.

Which is a contradiction.Hence (ii) holds.
(iii) is by similar method to part (ii).
(iv) Since by (i) either $\mu(0) \ge \mu(x)$ or $\beta(0) \ge \beta(x)$, for all $x\in X$.
Without loss of generality we may assume that $\beta(0) \ge \beta(x)$, for all $x\in X$, from (iii) it follows that either $\mu(0) \ge \mu(x)$ or $\mu(0) \ge \beta(x)$. If$\mu(0) \ge \beta(x)$, for any $x\in X$, then$(\mu\times\beta)(0,x) = \min\{\mu(0), \beta(x)\} = \beta(x)$. Let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X\times X$,
since $\mu\times\beta$ is a fuzzy KUS-ideal of $X\times X$, we have
$(\mu\times\beta)(z_1*x_1,z_2*x_2)\ge\min\{(\mu\times\beta)(z_1*y_1,z_2*y_2), (\mu\times\beta)(y_1*x_1,y_2*x_2)\}$--(A)
If we take $x_1 = y_1 = z_1 =0$, then
$\beta(z_2*x_2) = (\mu\times\beta)(0,z_2*x_2)$
$\ge \min\{(\mu\times\beta)(0,z_2*y_2),(\mu\times\beta)(0,y_2*x_2)\}$
$=\min\{\min\{\mu(0), \beta(z_2*y_2)\}, \min\{\mu(0), \beta(y_2*x_2)\}\}$
$= \min\{\beta(z_2*y_2), \beta(y_2*x_2)\}$

This prove that β is a fuzzy KUS-ideal of X . Now we consider the case
$\mu(0) \geq \mu(x)$ for all $x \in X$. Suppose that $\mu(0) < \mu(y)$, for some $y \in X$. then $\beta(0) \geq \beta(y) > \mu(0)$. Since $\mu(0) \geq \mu(x)$for all $x \in X$, it follows that $\beta(0) \geq \mu(x)$ for any $x \in X$. Hence $(\mu \times \beta)(x,0) = \min\{\mu(x), \beta(0)\} = \mu(x)$, taking $x_2 = y_2 = z_2 = 0$ in (A), then $\mu(z_1 * x_1) = (\mu \times \beta)(z_1 * x_1, 0)$
$\geq \min\{(\mu \times \beta)(z_1 * y_1, 0), (\mu \times \beta)(y_1 * x_1, 0)\} = \min\{\min\{\mu(z_1 * y_1), \beta(0)\}, \min\{\mu(y_1 * x_1), \beta(0)\}\} = \min\{\mu(z_1 * y_1), \mu(y_1 * x_1)\}$. Which proves that μ is a fuzzy KUS-ideal of X . Hence either μ or β is a fuzzy KUS-ideal of X . ⌂

Theorem 4.12. Let β be a fuzzy subset of a KUS-algebra X and let R_β be the strongest fuzzy relation on X , then β is a fuzzy KUS-ideal of X if and only if R_β is a fuzzy KUS-ideal of $X \times X$.

Proof: Assume that $\beta$is a fuzzy KUS-ideal of X . By proposition(4.7) ,we get, $R_\beta(0,0) \geq R_\beta(x,y)$, for any $(x,y) \in X \times X$.
Let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X \times X$, we have from ($Fkus_2$) :

$R_\beta(z_1 * x_1, z_2 * x_2) = \min\{\beta(z_1 * x_1), \beta(z_2 * x_2)\}$
$\geq \min\{\min\{\beta(z_1 * y_1), \beta(y_1 * x_1)\}, \min\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}\}$
$= \min\{\min\{\beta(z_1 * y_1), \beta(z_2 * y_2)\}, \min\{\beta(y_1 * x_1), \beta(y_2 * x_2)\}\}$
$= \min\{R_\beta(z_1 * y_1, z_2 * y_2), R_\beta(y_1 * x_1, y_2 * x_2)\}$
Hence R_β is a fuzzy KUS-ideal of $X \times X$.

Conversely, suppose that R_β is a fuzzy KUS-ideal of $X \times X$, by proposition (4.8)
$\beta(0) \geq \beta(x)$, for all $x \in X$, which prove ($Fkus_1$) .
Now, let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X \times X$, then
$\min\{\beta(z_1 * x_1), \beta(z_2 * x_2)\} = R_\beta(z_1 * x_1, z_2 * x_2)$
$\geq \min\{R_\beta((z_1,z_2) * (y_1, y_2)), R_\beta((y_1,y_2) * (x_1, x_2))\}$
$= \min\{R_\beta((z_1 * y_1),(z_2 * y_2)), R_\beta((y_1 * x_1),(y_2 * x_2))\}$
$= \min\{\min\{\beta(z_1 * y_1), \beta(z_2 * y_2)\}, \min\{\beta(y_1 * x_1), \beta(y_2 * x_2)\}\}$
In particular if we take $x_2 = y_2 = z_2 = 0$, then $\beta(z_1 * x_1) \geq \min\{\beta(z_1 * y_1), \beta(y_1 * x_1)\}$ .This proves ($Fkus_2$) and β is a fuzzy KUS-ideal of X .⌂

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The index

Algorithms for fuzzy KUS-ideal of KUS-algebra

Input (X : KUS-algebra, I : subset of X) ;

Output (“ I is a fuzzy KUS-ideal of X or not”) ;

```
Begin
If I = ∅then
Go to (1.) ;
End if
 If 0 ∉I then
 Go to (1.) ;
End if
Stop : = false
i : = 1;
 While i ≤ | x | and not (stop) do
 j : = 1
  While j ≤ | x | and not (stop) do
  k : = 1
   While k ≤ | x | and not (stop) do
   If (z_i *y_j) ∈ I and (y_j *x_k) ∈ I then
   If (z_i *x_k ) ∉ Ithen
   Stop : = false
   End if
  End while
 End while
End while
If stop then
Output ( “ I is a fuzzy KUS-ideal of X”)
Else
 (1.) output ( “ I is not a fuzzy KUS-ideal of X” )
End if

End.
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المثاليات الفازيةKUS في جبر-KUS

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الخلاصة

الهدف الرئيسي من هذا البحث هو اننا قدمنا مفهوم المثاليات الفازيةKUS في الجبر -KUS ، وتم برهنه العديد من النظريات والخصائص وعلاقة هذا الجبر بالجبريات الأخرى ووضعنا بعض الأمثلة عليها مستخدما برامج الحاسب الآلي .