Naturally Prime (Primary) Submodules

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Abstract

For a commutative ring (with identity) R and an R-module M, we introduce the notion of naturally prime modules, and naturally primary submodules (modules). We give some related results.

Introduction

Throughout this paper R denotes a commutative ring with identity, and M is an R-module. For submodules N, K of M, the product of N and K (denoted by $N\cdot K$) is defined as (N:M)(K:M)M[2]. A proper submodule N of M is called naturally prime if whenever K, H proper submodules of M such that $K\cdot H \subseteq N$, then either $K \subseteq N$ or $H \subseteq N$ [2]. A proper ideal I of a ring R is said to be naturally prime if it is a naturally prime R-submodule of R. Note that an ideal I is naturally prime equivalent to I is a prime ideal. We introduce the following: an R-module M is said to be naturally prime if the zero submodule of M is a naturally prime submodule. Also we say that a proper submodule N of M is naturally primary if whenever $K, H < M$, $K\cdot H \subseteq N$, then either

(1) $K \subseteq N$ or $H^n \subseteq N$ for some $n \in Z_+$

or

(2) $H \subseteq N$ or $K^n \subseteq N$ for some $n \in Z_+$.

An R-module M is called naturally primary if the zero submodule of M is naturally primary.

This paper consists of two sections, in section one we study naturally prime submodules (modules) and obtain some related results. In section two we study naturally primary submodules (modules) we give the basic properties about these concepts, also we give some relationships between naturally prime submodules (modules) and naturally primary submodules (modules).

1. Naturally Prime Submodules and Naturally Prime Modules

Recall that a proper submodule N of an R-module M is called prime if whenever $r \in R$, $x \in M$, $rx \in N$ implies either $x \in N$ or $r \in (N:M)$, where $(N:M) = \{r \in R: rM \subseteq N\}$,[2] .

It is clear that if N is a prime submodule of M, then (N:M) is a prime ideal of R.

Recall that an R-module M is called a multiplication R-module if for each $N \le M$, there exists an ideal I of R such that $N = IM$,[3].

Equivalently, M is a multiplication R-module if for each $N \le M$, $N = (N \underset{R}{:} M)M$,[3].

Proposition 1.1:[1]

Let M be an R-module, let $N < M$. If N is a natural prime submodule, then N is a prime submodule.

The converse is true if M is a multiplication R-module.

Example 1.2:[1]

Consider the Z-module $M = Z \oplus Z_p$, where p is a prime number, let $N = pZ \oplus Z_p$. N is a maxmal submodule, so it is a prime submodule. But

$(Z \oplus (0))^2 = (Z \oplus (0) \underset{Z}{:} Z \oplus Z_p)^2 (Z \oplus Z_p)$

$= p^2 Z\ (Z \oplus Z_p) = p^2 Z \oplus (0) \subseteq pZ + Z_p = N$

However, $Z \oplus (0) \not\subseteq pZ \oplus Z_p = N$. Thus N is not a naturally prime submodule of M.

The following remark is clear.

Remark 1.3:

Let M be an R-module, let $N < M$. Then N is a prime submodule of M if and only if $\{r \in R: \exists\ m \notin N \text{ and } rm \in N\} = (N:M)$.

Proposition 1.4:

Let M be a multiplication R-module and let $N < M$. Then the following statements are equivalent:

(1) N is a prime submodule of M

(2) N is a naturally prime submodule of M.

(3) $\{r \in R: \exists\ m \notin N \text{ and } rm \in N\} = (N:M)$.

Proof:

It follows by proposition 1.1 and remark 1.3.

Corollary 1.5:[1]

Let M be a faithful multiplication R-module. Then the following statements are equivalent

(1) M is a domain; that $Z_d(M) = \{r \in R: \exists\ m \in M, m \neq 0, rm = 0\}$

(2) (0) is a naturally prime submodule of M.

Proof:

By proposition 1.4, (0) is naturally prime if and only if $\{r \in R: \exists\ m \in M, m \neq 0, rm = 0\} = (0 \underset{R}{:} M)$. But M is faithful, so $(0 \underset{R}{:} M) = (0)$. Hence the result is obtained.

Proposition 1.6:

Let M be a multiplication R-module. Then the following statements are equivalent:

(1) N is a naturally prime submodule.

(2) N is a prime submodule.

(3) (N:M) is a prime ideal of R.

Proof:

(1) $\Leftrightarrow$ (2) (it follows by proposition 1.1)

(2) $\Leftrightarrow$ (3) see [4]

The following lemmas are needed for the next result.

Lemma 1.7:

Let $f: M \longrightarrow M'$ be an R-epimorphism, let $H \leq M'$. Then $(f^{-1}(H) \underset{R}{:} M) = (H \underset{R}{:} M')$.

Proof:

Let $r \in (f^{-1}(H) \underset{R}{:} M)$. Then $rM \subseteq f^{-1}(H)$ and so $f(rM) \subseteq ff^{-1}(H)$; that is $rM' \subseteq ff^{-1}(H) \subseteq H$. Thus $r \in (H:M')$ and so $(f^{-1}(H):M) \subseteq (H:M')$.

Conversely, let $r \in (H:M')$. Hence $rM' \subseteq H$. But $M' = f(M)$, so $rf(M) \subseteq H$; that is $f(rM) \subseteq H$. Thus for each $m \in M$, $f(rm) \in H$ and so $rm \in f^{-1}(H)$. This implies $r \in (f^{-1}(H):M)$.

Lemma 1.8:

Let $f: M \longrightarrow M'$ be an R-epimorphism, let $N \leq M$ such that $\ker f \subseteq N$. If $H, K \leq M$ such that $H \cdot K \subseteq f(N)$, then $f^{-1}(H) \cdot f^{-1}(K) \subseteq N$.

Proof:

If $H \cdot K \subseteq f(N)$, then $(H:M')(K:M')M' \subseteq f(N)$ and so by lemma 1.7, $(f^{-1}(H):M)(f^{-1}(K):M)M' \subseteq f(N)$ and since f is an epimorphism, we have $f\,[(f^{-1}(H):M)(f^{-1}(K):M)M] \subseteq f(N)$. It follows that $f\,[(f^{-1}(H) \cdot f^{-1}(K)] \subseteq f(N)$ and so that $f^{-1}(H) \cdot f^{-1}(K) \subseteq N$, because $\ker f \subseteq N$.

Theorem 1.9:

Let $f: M \longrightarrow M'$ be an R-epimorphism, let $N < M$ such that $\ker f \subseteq N$. If N is a naturally prime submodule of M, then f(N) is a naturally prime submodule of M′.

Proof:

Let H, K < M′ such that H·K $\subseteq$f(N). Then by lemma 1.8, $f^{-1}(H)\cdot f^{-1}(K) \subseteq N$. Since N is a naturally prime submodule of M, we have either $f^{-1}(H) \subseteq N$ or $f^{-1}(K) \subseteq N$. Hence $ff^{-1}(H) \subseteq f(N)$ or $ff^{-1}(K) \subseteq f(N)$ and so that either H $\subseteq$f(N) or K $\subseteq$f(N). Thus f(N) is naturally prime.

Remark 1.10:

The condition ker f $\subseteq$ N can't be dropped from theorem 1.9 as the following example shows:

The zero submodule of the Z-module Z is naturally prime. Let $\prod$: Z $\longrightarrow$ Z/6Z $\cong Z_6$, ker$\prod$ = 6Z $\not\subseteq$ (0) and $\prod(0) = (\bar{0})$ is not naturally prime submodule of Z_6.

Corollary 1.11:

Let M be an R-module, let N, W < M such that W $\supseteq$ N. If W is a naturally prime submodule of M, then $\frac{W}{N}$ is a naturally prime submodule of M.

Proof:

Let $\prod: M \longrightarrow \frac{M}{N}$ be the natural projection. It is clear that ker$\prod$ = N $\subseteq$ W. Hence the result follows by theorem 1.9.

Proposition 1.12:

Let M be a multiplication R-module, let N, W < M such that N $\subseteq$ W. If $\frac{W}{N}$ is a naturally prime submodule of $\frac{M}{N}$, then W is a naturally prime submodule of M.

Proof:

Since $\frac{W}{N}$ is naturally prime in $\frac{M}{N}$, $\frac{W}{N}$ is prime in $\frac{M}{N}$ and hence W is a prime submodule of M. But M is a multiplication R-module, so by proposition 1.1, W is a naturally prime submodule of M.

To obtain the next result, we need the following lemma.

Lemma 1.13:

Let f: M $\longrightarrow$M′be an R-epimorphism, let N < M. Then $(f(N) \underset{R}{:} M') \supseteq (N:M)$ and the reverse inclusion hold if ker f $\subseteq$ N.

Proof:

Let $r \in (N \underset{R}{:} M)$. Then rM$\subseteq$ N and rf(M) $\subseteq$ f(N); that is rM′$\subseteq$ f(N). Thus $r \in (f(N) \underset{R}{:} M')$ and hence $(N \underset{R}{:} M) \subseteq (f(N) \underset{R}{:} M')$. Now, let $r \in (f(N) \underset{R}{:} M')$ so that rM′$\subseteq$ f(N). Hence f(rM) $\subseteq$ f(N). Since ker f $\subseteq$ N, we get rM$\subseteq$ N. Thus $r \in (N \underset{R}{:} M)$. Therefore $(f(N) \underset{R}{:} M') \subseteq (N \underset{R}{:} M)$.

Proposition 1.14:

Let f: M $\longrightarrow$M′be an R-epimorphism such that ker f $\subseteq$ A, for each submodule A of M. If B < M′ and B is a naturally prime submodule of M′, then $f^{-1}(B)$ is a naturally prime submodule of M.

Proof:

Let N, W < M such that N·W $\subseteq f^{-1}(B)$. Hence $(N:M)(W:M)M \subseteq f^{-1}(B)$. It follows that $(N:M)(W:M)M' \subseteq ff^{-1}(B) \subseteq B$. Then by lemma 1.13, $(f(N):M')(f(W):M')M' \subseteq B$ and so f(N)·f(W) $\subseteq$ B. Since B is naturally prime in M′, we get either f(N) $\subseteq$ B or f(W) $\subseteq$ B. It follows that $N \subseteq f^{-1}f(N) \subseteq f^{-1}(B)$ or $W \subseteq f^{-1}f(W) \subseteq f^{-1}(B)$; that is $N \subseteq f^{-1}(B)$ or $W \subseteq f^{-1}(B)$. Thus $f^{-1}(B)$ ia a naturally prime submodule in M.

Recall that a submodule L of an R-module M is called strongly irreducible if whenever $L_1, L_2 \leq M$, $L \supseteq L_1 \cap L_2$, then $L \supseteq L_1$ or $L \supseteq L_2$, [5].

It is known that every prime submodule of a multiplication module is strongly irreducible. We shall prove that every naturally prime submodule is strongly irreducible, but first we prove the following lemma.

Lemma 1.15:

Let f: M be an R-module, let $N, W \leq M$. Then $N \cdot W \subseteq N \cap W$.

Proof:

$N \cdot W = (N \underset{R}{:} M)(W \underset{R}{:} M)M$

$\subseteq (N:M)W$

$\subseteq W$

Similarly $N \cdot W \subseteq N$.Thus $N \cdot W \subseteq N \cap W$.

Proposition 1.16:

Let N be a naturally prime submodule of an R-module M. Then N is strongly irreducible.

Proof:

Let $H, K \leq M$ such that $H \cap K \subseteq N$. Hence by lemma 1.15, $H \cdot K \subseteq N$. Since N is naturally prime, then either $H \subseteq N$ or $K \subseteq N$.

Proposition 1.17:

Let M be an R-module, let $N, K < M$ such that $K \not\subseteq N$. If N is a naturally prime submodule of M, then $N \cap K$ is a naturally prime of K.

Proof:

Let $A, B < K$ such that $A \cdot B \subseteq N \cap K$. Then $A \cdot B \subseteq N$. Since N is naturally prime, either $A \subseteq N$ or $B \subseteq N$. Thus $A \subseteq N \cap K$ or $B\ N \cap K$ because $A < K$, $B < K$.

Now we introduce the following:

Definition 1.18:

An R-module M is called naturally prime if (0) is a naturally prime submodule.

Hence a ring R is a naturally prime if and only if R is an integral domain.

Also by proposition 1.1, every natural prime module is a prime module.

Proposition 1.19:

Let M be a multiplication R-module. Then the following statements are equivalent:

(1) M is a naturally prime module.
(2) M is a prime module.
(3) M is a domain, i.e. $\{r \in R: \exists\, m \in M, m \neq 0, rm = 0\} = (0)$.
(4) annM is a prime ideal.

Proof:

It follows by proposition 1.6, corollary 1.5.

The following remark is easy:

Remark 1.20:

Let M, M′ be R-isomorphic modules. Then M is naturally prime if and only if M′ is naturally prime.

Now by the same example 1.10, we get that a homomorphic image of naturally prime module need not naturally prime module.

The last result in this section is the following:

Proposition 1.21:

Let N be a naturally prime submodule of an R-module M. Then $\frac{M}{N}$ is a naturally prime R-module.

Proof:

Let $\prod: M \longrightarrow \frac{M}{N}$ be the natural projection. By theorem 1.9, $\prod(N) = 0_{\frac{M}{N}}$ is a naturally prime in $\frac{M}{N}$. Thus $\frac{M}{N}$ is a naturally prime module.

2. Naturally Primary Submodules (Modules)

In this section we introduce the notions of naturally primary submodule, naturally primary module. We investigate the basic properties related with these concepts. Also we give some relationships between these concepts and the concepts of naturally prime submodule and naturally prime module.

Recall that a proper submodule N of an R-module M is called primary if whenever $r \in R$, $x \in M$, $rx \in N$ implies $x \in N$ or $r^n \in (N:M)$ for some $n \in Z_+$, [6]. An R-module is called primary if (0) (zero submodule) is a primary submodule, [7].

We introduce the following:

Definition 2.1:

A proper submodule N of an R-module M is called naturally primary if whenever H, K < M such that $H \cdot K \subseteq N$, then either

(1) $H \subseteq N$ or $K^n \subseteq N$ for some $n \in Z_+$, or

(2) $K \subseteq N$ or $H^n \subseteq N$ for some $n \in Z_+$.

Definition 2.2:

An R-module M is called naturally primary if (0) (the zero submodule of M) is a naturally primary submodule.

Remarks 2.3:

(1) A naturally primary submodule need not be primary as for example:

The zero submodule, (0) of the Z-module Z_{p^∞} is not primary submodule. But for each A, B < Z_{p^∞}. If $A \cdot B = (0)$. Suppose $A \neq (0)$, then for each n > 1 $B^n = (B: Z_{p^\infty})^n Z_{p^\infty} = 0 \cdot Z_{p^\infty} = (0)$. Thus (0) is naturally primary.

Note that we deduce (from this example) a naturally primary module need not primary.

(2) It is clear that a naturally prime submodule is naturally primary. But the converse is not true in general, since (0) submodule of the Z-module Z_{p^∞} is naturally primary (see (1)) but (0) is not naturally prime because it is not prime.

Proposition 2.4:

If N is a primary submodule of a multiplication R-module M, then N is naturally primary.

Proof:

Let A, B < M such that $A \cdot B \subseteq N$. Hence $(A:M)(B:M)M \subseteq N$. Since M is a multiplication R-module, $(A:M)B \subseteq N$. But N is primary, so either $B \subseteq N$ or $(A:M)^n \subseteq (N:M)$ for some $n \in Z_+$. Hence either $B \subseteq N$ or $A^n = (A:M)^n M \subseteq (N:M)M = N$. Thus B is naturally primary.

It follows by proposition 2.4, every multiplication primary module is naturally primary.

Now we shall show that under certain class of modules, naturally primary submodules (modules) and naturally prime submodules (modules) are equivalent.

Proposition 2.5:

Let M be a multiplication R-module and let N < M such that $(N \underset{R}{:} M)$ is a semiprime ideal of R. If N is a naturally primary submodule of M, then N is naturally prime.

Proof:

Let A, B < M such that $A \cdot B \subseteq N$. Since N is naturally primary, there are two possibilities
(1) $A \subseteq N$ or $B^n \subseteq N$ for some $n \in Z_+$.
(2) $B \subseteq N$ or $A^n \subseteq N$ for some $n \in Z_+$.
For case (1): $A \subseteq N$ or $B^n \subseteq N$ for some $n \in Z_+$. Hence $A \subseteq N$ or $(B:M)^n M \subseteq N$ for some $n \in Z_+$; that is $A \subseteq N$ or $(B:M)^n \subseteq (N:M)$. Thus either $A \subseteq N$ or $(B:M) \subseteq \sqrt{(N:M)}$. But $\sqrt{(N \underset{R}{:} M)} = (N:M)$, since (N:M) is semiprime, so that either $A \subseteq N$ or $(B:M) \subseteq (N:M)$. Thus either $A \subseteq N$ or $B = (B:M)M \subseteq (N:M)M = N$.
Similarly case (2) implies either $A \subseteq N$ or $B \subseteq N$. Therefore N is a naturally prime submodule.

Corollary 2.6:
Let M be a multiplication R-module and let N < M such that $(N \underset{R}{:} M)$ is a semiprime. Then the following statements are equivalent:
(1) N is a naturally primary submodule of M.
(2) N is a naturally prime submodule of M.
(3) N is a prime submodule of M.
(4) (N:M) is a prime ideal of M.

Corollary 2.7:
Let M be a multiplication R-module. Then the following statements are equivalent:
(1) M is a naturally primary module.
(2) M is a naturally prime module.
(3) M is a prime module.
(4) $\text{ann}_R M$ is a prime ideal of R.

Recall that an R-module is called fully idempotent if every submodule N of M is idempotent; that $N = N^2 = (N:M)^2 M$, [8].

Proposition 2.8:
Let M be a fully idempotent, let N < M then N is a naturally primary submodule if and only if N is a naturally prime submodule.
Proof:
($\Rightarrow$) Let A, B < N such that $A \cdot B \subseteq N$. Since N is naturally primary submodule, there are two cases
(1) $A \subseteq N$ or $B^n \subseteq N$ for some $n \in Z_+$.
(2) $B \subseteq N$ or $A^n \subseteq N$ for some $n \in Z_+$.
For case (1): Since M is fully idempotent, $B^2 = B$ and hence $B^n = B$, for each $n \in Z_+$. Thus (1) implies $A \subseteq N$ or $B \subseteq N$.
Similarly case (2) implies $A \subseteq N$ or $B \subseteq N$. Thus N is naturally prime.
($\Leftarrow$) It follows by remark 2.3(2).

Corollary 2.9:
Let M be a fully idempotent R-module and let N < M. Then the following statements are equivalent:
(1) N is a naturally primary.
(2) N is a naturally prime.
(3) N is a prime.
(4) N is a primary.
Proof:
(1) $\Leftrightarrow$ (2) It follows by proposition 2.8.
(2) $\Rightarrow$ (3) It follows by proposition 1.1.
(3) $\Rightarrow$ (4) It is clear.
(4) $\Rightarrow$ (1) Since M is fully idempotent, M is multiplication [8], hence the result follows by proposition 2.4.

Hence by corollary 2.9, if M is fully idempotent, then M is a naturally primary module if and only if M is naturally prime if and only if M is prime, if and only if M is primary.
Recall that an R-module M is called fully pure if every submodule N of M is pure [8], where a submodule N of M is pure if $IM \cap N = IN$ for each ideal I of R, [9].

Some authors use the name regular module for “fully pure module”.

Lemma 2.10:

Let M be a multiplication fully pure R-module, then $N\cdot W = N \cap W$.

Proof:

$N \cap W = (N:M)M \cap W$, since M is a multiplication R-module

$= (N:M)W$, since M is fully pure

$= (N:M)(W:M)M$, since M is fully pure

$= N\cdot W$

By [8], proposition 2.6, a fully idempotent module is equivalent to $N\cdot W = N \cap W$ for each $N, W \leq M$. Hence every multiplication fully pure is fully idempotent and so we get the following result.

Proposition 2.11:

Let M be a multiplication fully pure R-module, let $N < M$. Then N is a naturally prime submodule if and only if N is a naturally primary submodule.

Hence by proposition 2.11, every multiplication fully pure module is prime if and only if it is primary.

Now we turn attention for the image and inverse image of primary submodules.

Theorem 2.12:

Let $f: M \longrightarrow M'$ be an epimorphism let $N < M$ such that $\ker f \subseteq N$. If N is a naturally primary submodule of M, then f(N) is a naturally primary submodule of M′.

Proof:

Let $H, K < M'$ such that $H\cdot K \subseteq f(N)$. Then by lemma 1.8, $f^{-1}(H)\cdot f^{-1}(K) \subseteq N$ and since N is naturally primary, we have two cases:

(1) $f^{-1}(H) \subseteq N$ or $(f^{-1}(K))^n \subseteq N$ for some $n \in Z_+$.

(2) $f^{-1}(K) \subseteq N$ or $(f^{-1}(H))^n \subseteq N$ for some $n \in Z_+$.

Consider case (1), $f^{-1}(H) \subseteq N$ or $(f^{-1}(K))^n \subseteq N$ for some $n \in Z_+$. If $f^{-1}(H) \subseteq N$, then $ff^{-1}(H) \subseteq f(N)$. But $H = ff^{-1}(H)$, so $H \subseteq f(N)$.

If $(f^{-1}(K))^n \subseteq N$, then $(f^{-1}(K):M)^n M \subseteq N$ and so by lemma 1.7 $(K:M')^n M \subseteq N$. It follows that $(K:M')^n M' \subseteq f(N)$; that is $K^n \subseteq f(N)$.

Similarly case (2), implies either $K \subseteq f(N)$ or $H^n \subseteq f(N)$ for some $n \in Z_+$. Thus f(N) is a naturally primary submodule of M.

Corollary 2.13:

Let $N, W < M$ such that $W \supseteq N$ if W is a naturally primary submodule of M, then $\frac{W}{N}$ is a naturally primary submodule of $\frac{M}{N}$.

Proof:

Let $\prod : M \longrightarrow \frac{M}{N}$ be the natural projection. Hence by theorem 2.12, the result follows.

Note that the converse of corollary 2.13 holds in the class of multiplication modules. However, first we need the following lemma.

Lemma 2.14:

Let M be a multiplication R-module, let $A, B \leq M$. Then $A^n + B = (A + B)^n + B$, for each $n \in Z_+$.

Proof:

The proof is by induction, if $n = 1$, the result is clear. If $n = 2$, then

$(A + B)^2 = (A + B)\cdot(A + B)$

$= A\cdot(A + B) + B\cdot(A + B)$ by [1]

$= A^2 + A\cdot B + B\cdot A + B^2$

$= A^2 + A\cdot B + A\cdot B + B^2$ since $\cdot$ is commutative

Hence $(A + B)^2 + B = A^2 + A\cdot B + A\cdot B + B^2 + B$

$= A^2 + B$ since $AB \subseteq B$, $B^2 \subseteq B$.

Now assume $(A + B)^k + B = A^k + B$. To prove $(A + B)^{k+1} + B = A^{k+1} + B$

$(A + B)^{k+1} + B = (A + B)^k\cdot(A + B) + B$
$= (A^k + B)(A + B) + B$
$= A^{k+1} + BA + AB + B^2 + B$
$= A^{k+1} + B$

Theorem 2.15:

Let M be a multiplication R-module, N, M < M with $N \subseteq W$. If $\frac{W}{N}$ is a naturally primary submodule of $\frac{M}{N}$, then W is a naturally prime submodule of M.

Proof:

Let A, B < M such that $A\cdot B \subseteq W$. Hence $\frac{A\cdot B+N}{N} \subseteq \frac{W}{N}$. We claim that $\frac{(A+N)\cdot(B+N)+N}{N} \subseteq \frac{AB+N}{N}$. To show this

$$\frac{(A+N)}{N}\cdot\frac{(B+N)}{N} = \left(\frac{A+N}{N} \underset{R}{:} \frac{M}{N}\right)\left(\frac{B+N}{N} \underset{R}{:} \frac{M}{N}\right)\frac{M}{N}$$
$$= \left(A+N \underset{R}{:} M\right)\left(B+N \underset{R}{:} M\right)\frac{M}{N}$$
$$= \frac{(A+N \underset{R}{:} M)(B+N \underset{R}{:} M)M+N}{N}$$
$$= \frac{(A+N)\cdot(B+N)+N}{N}$$

But $(A + N)\cdot(B + N) = (A + N)B + (A + N)N$ (1)

$= AB + NB + AN + N^2$

$\subseteq AB + N$

Thus $\frac{A+N}{N}\cdot\frac{B+N}{N} \subseteq \frac{AB+N}{N} \subseteq \frac{W}{N}$. Since $\frac{W}{N}$ is a naturally primary submodule of $\frac{M}{N}$, there are two possibilities:

(1) $\frac{A+N}{N} \subseteq \frac{W}{N}$ or $\left(\frac{B+N}{N}\right)^n \subseteq \frac{W}{N}$ for some $n \in Z_+$.

(2) $\frac{B+N}{N} \subseteq \frac{W}{N}$ or $\left(\frac{A+N}{N}\right)^n \subseteq \frac{W}{N}$ for some $n \in Z_+$.

(1) If $\frac{A+N}{N} \subseteq \frac{W}{N}$, then $A + N \subseteq W$. Hence $A \subseteq W$.

If $\left(\frac{B+N}{N}\right)^n \subseteq \frac{W}{N}$. Since

$$\left(\frac{B+N}{N}\right)^n = \left(\frac{B+N}{N} : \frac{M}{N}\right)^n \frac{M}{N} = (B+N:M)^n \frac{M}{N}$$

$$= \frac{(B+N:M)^n M+N}{N} = \frac{(B+N)^n M+N}{N}$$

$= \frac{B^n+N}{N}$ by lemma 2.14

Hence $\frac{B^n+N}{N} \subseteq \frac{W}{N}$ and so $B^n \subseteq W$.

Similarly (2), implies $B \subseteq W$ or $A^n \subseteq W$. Thus W is a naturally primary submodule of M.

Before giving our next result, we prove the following lemma:

Lemma 2.16:

Let $f: M \longrightarrow M'$ be an epimorphism let A < M such that ker $f \subseteq A$.Then$f(A^n) = (f(A))^n$ for each $n \in Z_+$.

Proof: For each $n \in Z_+$,

$f(A^n) = f\,[(A:M)^n M]$
$= (A:M)^n M'$
$= (f(A):M')^n M'$ by lemma 1.13
$= (f(A))^n$

Proposition 2.17:

Let $f: M \longrightarrow M'$be an epimorphism such that ker $f \subseteq A$, for each $A \leq M$. If W is a naturally primary submodule of M′, then $f^{-1}(W)$ is a naturally primary submodule of M.

Proof:

Let A, B < M such that $A\cdot B \subseteq f^{-1}(W)$. Then $(A:M)(B:M)M \subseteq f^{-1}(W)$ and so $(A:M)(B:M)M' \subseteq ff^{-1}(W) \subseteq W$. By lemma 1.13, $(A:M) = (f(A):M'),(B:M) = (f(B):M')$. Thus $(f(A):M')(f(B):M')M' \subseteq W$; that is $f(A)\cdot f(B) \subseteq W$. But W is naturally primary submodule in M′, so there are 2-cases

(1) $f(A) \subseteq W$ or $(f(B))^n \subseteq W$ for some $n \in Z_+$.

(2) $f(B) \subseteq W$ or $(f(A))^n \subseteq W$ for some $n \in Z_+$.

For case (1):

If $f(A) \subseteq W$, then $f^{-1}f(A) \subseteq f^{-1}(W)$. Hence $A \subseteq f^{-1}(W)$.

If $(f(B))^n \subseteq W$. By lemma 2.16, $(f(B))^n = f(B^n)$. So $f(B^n) \subseteq W$ and this implies $f^{-1}f(B^n) \subseteq f^{-1}(W)$. Thus $B^n \subseteq f^{-1}(W)$.

Similarly case (2), implies either $B \subseteq f^{-1}(W)$ or $A^n \subseteq f^{-1}(W)$. Therefore $f^{-1}(W)$ is naturally primary.

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