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*n-normal weighted composition operator and n-unitary weighted composition operator on Hardy space H^2

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Abstract

In this paper we characterize the n-normal weighted composition operator and n-unitary weighted composition operator on Hardy space H^2

Keyword

Composition operator, weighted composition operator, normal operator, Hardy space .

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1-Introduction

Let U denote the open unite disc in the complex plan ,let H denote the collection of all holomorphic function on U and let H^2 is consisting of all holomorphic self-map on U such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e)

$\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

More precisel $f(z) = \sum_{n=0}^{\infty} a_n z^n$y if and only if $\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty$.The inner product inducing on H^2 is define as follows $\langle f,g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$ for all $f.g$ in H^2 with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$

If $f \in H$and φ is analytic self map of the unit disk U ,the weighted composition operator onH^2 is defined by $W_{f,\varphi} = T_f C_\varphi$,where T_f is the Toeplitz operator $T_f : H^2 \to H^2$

Defined by $T_f(h) = fh$ for h inH^2 and C_φ is the composition operator on H^2 given by $C_\varphi(h) = ho\varphi$,if f is bounded on the unit disk U then $w_{f,\varphi}$ is bounded on H^2 and $\|W_{f,\varphi}\| = \|T_f C_\varphi\| \le \|f\|_\infty \|C_\varphi\|$

2- Preliminaries

Theorem (2-1) [5]

The composition operator C_φ is an identity operator if and only if φ is the identity self-map

Theorem (2-2) [5]

Let φ and γ be holomorphic self – maps $then\ C_\varphi C_\gamma = C_{\gamma o \varphi}$

For each $\alpha \in U$,the reproducing kernel at α ,denoted by $K_\alpha(z)$ and defined by Shapiro [5] as follows $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$

It is easily seen for each $\alpha \in U$ and $f \in H^2, f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$\langle f,K_\alpha \rangle = \sum_{n=0}^{\infty} a_n\, \alpha^n = f(\alpha)$

Theorem(2-3) [5]

Let φ be a holomorphic self-map of U .then for all α in U $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$

Recall that for $f \in H$ the Toeplitz operator T_f is the operator $T_f : H^2 \to H^2$ and given by $T_f(h) = fh$ for h inH^2 ,and it have the properties [4]

(a) $C_\varphi T_f = T_{fo\varphi}\, C_\varphi$
(b) $T_g T_f = T_{gf}$
(c) $T_{f+\gamma g} = T_f + T_{\gamma g}$
(d) $T_f{}^* = T_{\bar{f}}$

Theorem(2-4)

Suppose that $W_{f,\varphi}: H^2 \to H^2$ is bounded and $\alpha \in U$ then $W_{f,\varphi}^* K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}$

When $\varphi(z) = (az+b)/cz+d)$ is anon linear fractional self mapping of U,Cowen [1] establishes $C_\varphi^* = T_g C_\sigma T_h^*$,where the Cowen auxiliary functions g, σ and h are defined as follows :

$g(z) = \frac{1}{-\bar{b}z+\bar{d}}$,$\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ $and\ \ h(z) = cz+d$.

If φ is linear fractional and the weight function *f* is both linear fractional and bounded on U then $W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^*$

Recall that the operator S is normal if $S^*S = SS^*$ where S^* is the adjoint of S and is said to be unitary if $S^*S = SS^* = I$ where I is the identity operator on H [9]

Theorem(2-5)[9]

If S is normal operator then $\|S^*f\| = \|Sf\|$ for all $f \in H$

Theorem(2-6)[9]

If S is normal operator then τ is an eigenvalue for S if and only if $\bar{\tau}$ is is an eigenvalue forS^*

Theorem (2-7)[5]

Let φ be a holomorphic self-map .then C_φ is normal if and only if $\varphi = \alpha z$,

For some $|\alpha| \le 1$

Definition (2-8)[8]

$An\ operator\ S \in B(H)$ is called an n-normal operator if $S^*S^n = S^nS^*$

Theorem(2-9)[8]

Let $S \in B(H)$ then S is n-normal operator if S^n is normal where $n \in \mathbb{N}$

Theorem(2-10)[8]

Let $S \in B(H)$ be n-normal operator, then

1-S^* is n-normal

2- If S^{-1} exists then S^{-1} is n-normal

3- If $R \in B(H)$ is unitary equivalent to S ,then R is n-normal

Definition (2-11)[7]

Let φ aholomorphic self-map of U , $\varphi$is called an inner function if $|\varphi(z)| = 1$ almost every where on ∂U

Theorem(2-12) [7]

If φ is univalent inner function then φ is an automorphism of U

Theorem(2-13) [7]

Suppose that ϕ is not elliptic automorphism, then there is a unique fixed point p of ϕ with $|p| \leq 1$ *and* $|\phi'(p)| \leq 1$.

By Denjoy-Wolff theorem, the fixed point p in Grand Iteration theorem to which the iterates of ϕ converges is unique and it is called the Denjoy-Wolff point of ϕ or attractive fixed point for ϕ.

3- The Main results

Definition (3-1)

The operator $S \in B(H)$ is called an n-unitary operator if $S^*S^n = S^nS^* = I$

Proposition: (3-2)

Let φ be a holomorphic self-map .then C_φ is n- normal if and only if $\varphi = \gamma z$,

For some $|\gamma| \leq 1$

Proof :

Suppose C_φ is n- normal since by theorem (2-2) $C_\varphi^n = C_{\varphi_n}$

And since by theorem (2-9) we have C_{φ_n} is normal

Then by theorem (2,7)$\varphi = \gamma z$

Conversely suppose $\varphi = \gamma z$

Then $\varphi_n = \varphi o \varphi o \varphi \dots o \varphi$

$$= \gamma(\gamma(\gamma(\dots\gamma(\gamma(z)) = \gamma^n z$$

Take $\gamma^n = \beta$ so $\varphi_n = \beta z$

Then $C_{\varphi_n} = C_\varphi^n$ is normal, hence C_φ is n-normal.

Corollary : (3-3)

Let φ be a holomorphic self-map .then C_φ is n-normal if and only if C_φ is normal

Proof:

we have the result by theorem(2-2), theorem(2-9) and proposition(3-2)

Proposition: (3-4)

Let φ be a holomorphic self-map .then C_φ is n- unitary if and only if $\varphi_n = \beta z$,

For some $|\beta| \leq 1$

Proof:

$$C_\varphi^n C_\varphi^* K_{\alpha(z)} = K_\alpha(z)$$

$$C_\varphi^n [illegible] \alpha(z) = C_{\varphi_n} K_{\varphi(\alpha)}(z) = \frac{[illegible]}{1-[illegible])\varphi_n(z)}$$

$$\frac{[illegible]}{1-[illegible])\varphi_n(z)} = \frac{1}{1-\bar{\alpha}z}$$

$$[illegible]\overline{\varphi(\alpha)}\varphi_n(z) = 1 - \bar{\alpha}z$$

$$\overline{\varphi(\alpha)}\varphi_n(z[illegible] \bar{\alpha}z$$

$$\varphi_n(z) = (\frac{\bar{\alpha}}{\overline{\varphi(\alpha)}})z \text{ [illegible]t } \beta = \frac{[illegible]}{\varphi[illegible])}$$

Since $\bar{\alpha} \leq 1$ and [illegible]) ≤ 1 hence $|\beta| \leq 1$

On the other hand

$$C_\varphi^* C_\varphi^n K_{\alpha(z)} = K_\alpha(z)$$

$$C_\varphi^* C_\varphi^n K_{\alpha(z)} = C_\varphi^* C[illegible]K_\alpha(z) = C_\varphi^* K_\alpha(\varphi_n(z)) = K_{\varphi(\alpha)}(\varphi_n(z)) = \frac{[illegible]}{1-[illegible])\varphi_n(z)}$$

$$\frac{1}{1-\overline{\varphi(\alpha)}\varphi_n(z)} = \frac{1}{1-\bar{\alpha}z}$$

$$1-\overline{\varphi(\alpha)}\varphi_n(z) = 1-\bar{\alpha}z$$

$$\overline{\varphi(\alpha)}\varphi_n(z) = \bar{\alpha}z$$

$$\varphi_n(z) = \left(\frac{\bar{\alpha}}{\overline{\varphi(\alpha)}}\right)z \text{ let } \beta = \frac{\bar{\alpha}}{\overline{\varphi(\alpha)}}$$

Since $\bar{\alpha} \leq 1$ and $\overline{\varphi(\alpha)} \leq 1$ hence $\varphi_n = \beta z$, $|\beta| \leq 1$

Proposition: (3-5)

Let $\varphi_1, \varphi_2, \dots \varphi_n$ be a holomorphic self-maps and $f_1, f_2, \dots f_n \in H$ then

$$W_{f_1\varphi_1} . W_{f_2\varphi_2} \dots . W_{f_n\varphi_n} = T_{f_1 . (f_2 o \varphi_1) . (f_3 o \varphi_2 o \varphi_1) \dots (f_2 o \varphi_{n-1} o \varphi_{n-2} o \dots o \varphi_1)} . C_{\varphi_n o \varphi_{n-1} o \dots o \varphi_1}$$

Proof :

$$W_{f_1\varphi_1} . W_{f_2\varphi_2} . W_{f_3\varphi_3} \dots . W_{f_n\varphi_n} = T_{f_1} C_{\varphi_1} . T_{f_2} C_{\varphi_2} . T_{f_3} C_{\varphi_3} . \dots . T_{f_n} C_{\varphi_n}$$

$$= T_{f_1} . T_{f_2 o \varphi_1} C_{\varphi_1} C_{\varphi_2} . T_{f_3} C_{\varphi_3} . \dots . T_{f_n} C_{\varphi_n}$$

$$= T_{f_1} . T_{f_2 o \varphi_1} C_{\varphi_2 o \varphi_1} . T_{f_3} C_{\varphi_3} . \dots . T_{f_n} C_{\varphi_n}$$

$$= T_{f_1} . T_{f_2 o \varphi_1} T_{f_3 o \varphi_2 o \varphi_1} \; C_{\varphi_3 o \varphi_2 o \varphi_1} . T_{f_4} C_{\varphi_4} . \dots . T_{f_n} C_{\varphi_n}$$

$$= T_{f_1} . T_{f_2 o \varphi_1} T_{f_3 o \varphi_2 o \varphi_1} \; T_{f_4 o \varphi_3 o \varphi_2 o \varphi_1} \; C_{\varphi_4 o \varphi_3 o \varphi_2 o \varphi_1} . T_{f_5} C_{\varphi_5} . \dots . T_{f_n} C_{\varphi_n}$$

$$= T_{f_1 . (f_2 o \varphi_1) . (f_3 o \varphi_2 o \varphi_1) \dots (f_2 o \varphi_{n-1} o \varphi_{n-2} o \dots o \varphi_1)} . C_{\varphi_n o \varphi_{n-1} o \dots o \varphi_1}$$

Corollary : (3-5)

Let φ be a holomorphic self-map and $f \in H$ then

$$W_{f,\varphi}^n = T_{f \; .(f \; o\varphi \;).(f \; o\varphi_2) \dots (f \; o\varphi_{n-1})} . C_{\varphi_n}$$

Proposition: (3-6)

If φ is non-constant holomorphic self-map on U and $W_{f,\varphi}$ is n-normal then either *f=0* or *f* never vanishes on U

Proof :

suppose $W_{f,\varphi}$ is n- normal and $f(\beta) = 0$ for some $\beta \in U$

Then $W_{f,\varphi}^* K_\beta = f(\beta) K_{\varphi(\beta)}$=0

Since $W_{f,\varphi}$ is n-normal then by theorem (2-2) ${W_{f,\varphi}}^n$ is normal and by theorem (2-5)

We have $\left\|(W_{f,\varphi}^n)^* K_\beta\right\| = \left\|(W_{f,\varphi}^n) \quad K_\beta\right\| = \left\|(W_{f,\varphi}^*)^n K_\beta\right\| = 0$

$(W_{f,\varphi}^n) \quad K_\beta = 0 \quad \forall \beta \in U$ then $(W_{f,\varphi}^n) \quad = 0$ since the $\{K_\beta\}_{\beta \in U}$ spanH^2

$$T_{f \; .(f \; o\varphi \;).(f \; o\varphi_2) \dots (f \; o\varphi_{n-1})} . C_{\varphi_n} = 0$$

Since $C_{\varphi_n} \neq 0$ then $f \;\; .(f \;\; o\varphi_1) . (f \;\; o\varphi_2) . \dots (f \;\; o\varphi_{n-1}) = 0$

it follows that either *f=0* or $(f \;\; o\varphi_1)$ … or $(f \;\; o\varphi_{n-1})$on non-empty open sub set $\varphi(U)$ or $\varphi(\varphi(U))$ …. Or $\varphi(\varphi_{n-2}(U))$ but φ is non-constant function on U

hence by open mapping theorem we have either *f=0* or *f* never vanishes on U

Proposition: (3-7)

Suppose $W_{f,\varphi}$ is n-normal. If φ is non-constant holomorphic self-map on U and f is not zero function then φ is univalent

Proof :

suppose φ is not univalent on U then there exists $a, b \in U$ such that $a \neq b$ and $\varphi(a) = \varphi(b)$

Since $f \neq 0$ then from proposition(3-6) we have $f(a) \neq 0$ and $f(b) \neq 0$

Let $g = \frac{K_a}{f(a)} - \frac{K_b}{f(b)}$ be non-zero function onH^2

$$W_{f,\varphi}{}^* g = W_{f,\varphi}{}^*\left(\frac{K_a}{f(a)} - \frac{K_b}{f(b)}\right)$$

$$= \frac{1}{f(a)} W_{f,\varphi}{}^* K_a - \frac{1}{f(b)} W_{f,\varphi}{}^* K_b$$

$$= \frac{1}{f(a)} \overline{f(a)} K_{\varphi(a)} - \frac{1}{f(b)} \overline{f(b)} K_{\varphi(b)}$$

$$= K_{\varphi(a)} - K_{\varphi(b)}$$

$$= \frac{1}{1-\overline{\varphi(a)}(z)} - \frac{1}{1-\overline{\varphi(b)}(z)} = 0$$

since $\varphi(a) = \varphi(b)$

We have $W_{f,\varphi}{}^* g = 0$

Since $W_{f,\varphi}$ is n-normal then $W_{f,\varphi}{}^n$ is normal

Then$\left\|(W_{f,\varphi}^n)\ \ g\right\| = \left\|(W_{f,\varphi}^n)^* g\right\| = \left\|(W_{f,\varphi}^*)^n g\right\| = 0$

If $\left\|(W_{f,\varphi}^n)\ \ g\right\| = 0$ we have $(W_{f,\varphi}^n)\ \ g = 0$

Then $T_{f\ .(f\ \ o\varphi\ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1})}.C_{\varphi_n} g = 0$

$f(z)\ \ .(f\ \ (\varphi(z\ \ )).(f\ \ (\varphi_2(z)).\dots(f\ \ (\varphi_{n-1}(z)\ g o\varphi_n(z) = 0$

Since $f \neq 0$

Then $f(z) \neq 0, f(\varphi(z)) \neq 0$,... and $f(\varphi_{n-1}(z)) \neq 0$

We have $g(\varphi_n(z)) = 0$

Since φ is non-constant then φ_n is non-constant

Since $\varphi(U) \subseteq U$ and $\varphi(U)$ is open in U by open mapping theorem we have

$g(\varphi_n(z)) = 0\ \ \forall z \in U$

Then *g=0* which is contradiction

Then φ is univalent

Proposition: (3-8)

If $W_{f,\varphi}$ be n-unitary then φ is an automorphism of U.

Proof:

Suppose $W_{f,\varphi}$ be n-unitary then $W_{f,\varphi}$ is norm preserving i.e $\|W_{f,\varphi} g\| = \|g\|$

Let g(z)=1

$$\left\|(W_{f,\varphi}^n)\ \ g\right\| = \left\|(W_{f,\varphi}^n)\ \ 1\right\| = \|1\| = 1$$

$$\left\|T_{f\ .(f\ \ o\varphi\ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1})}.C_{\varphi_n} 1\right\| = 1$$

$$\|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\| = 1$$

Again let g(z)=z , $\|g\| = 1$

$$1 = \|g\| = \left\|(W_{f,\varphi}^n)\ \ g\right\|$$

$$= \left\|T_{f\ .(f\ \ o\varphi\ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1})}.C_{\varphi_n} g\right\|$$

$$= \|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\ .g o\varphi_n\|$$

$$= \|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\ .g(\varphi_n)\|$$

$$= \|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\ .\varphi_n\|$$

Since $|\varphi_n(e^{it})| \leq 1$ a.e t$\in [0,2\pi]$

And both $\|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\| = 1$ *and*

$$\|f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2).\dots(f\ \ o\varphi_{n-1})\ .\varphi_n\| = 1$$

$|\varphi_n(e^{it})| = 1$ a.e on U

then φ_n is inner function and from proposition ( ) then φ_n is automorphism

hance $\varphi_n(z) = \gamma\alpha_p(z)$

$\varphi(\varphi_{n-1}(z)) = \gamma\alpha_p(z)$

φ is automorphism

Proposition: (3-9)

Suppose $\varphi(\beta) = 0$ $for\ some\ \beta \in U$,if $W_{f,\varphi}$ be n-unitary then

$$f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1}) = \ldots = \frac{K_\beta}{\|K_\beta\|}$$

Proof :

let $W_{f,\varphi}$ is n-unitary

Then $\ldots W^*_{f,\varphi}K_\beta = K_\beta$

$W^n_{f,\varphi} \ldots K_{\varphi(\beta)} = K_\beta$

Since $\ldots) = 0$ then $W^n_{f,\varphi} \ldots K_0 = K_\beta$ $and\ we\ have\ W^n_{f,\varphi} \ldots 1 = K_\beta$

$\ldots\overline{\beta}).f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})C_{\varphi_n}1 = K_\beta$

Hence

$$f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1}) = \ldots$$

Proposition: (3-10)

[illegible]unitary on H^2 if and only if φ_n is an [illegible]orphism o[illegible] and

$\ldots_n)(z)[\frac{(\ldots-|\alpha|^2)-\beta(1-\bar{c})z}{[(1-\overline{\rho\alpha}\omega\ldots)-(\bar{\beta}-\overline{\rho\alpha}\omega)z]^2}] = \ldots\ [\frac{(1-\ldots|^2)}{(1-\bar{\alpha}\rho z)[(1-\ldots)-\bar{\rho}(\bar{\alpha}-\bar{\beta})z]}]$

Proof:

Suppose $W_{f,\varphi}$ is n-unitary

Then by Proposition (3-8) φ_n must be an automorphism and by Proposition (3-9)

$$f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1}) = \ldots$$

Conversely

Since φ_n is an automorphism , $\varphi_n = \omega\frac{\beta-z}{1-\bar{\beta}z}$, $|\omega| = 1$ for some $\omega \in \partial U, \beta \in U$

Then φ must be automorphism $\varphi = \rho\frac{\alpha-z}{1-\bar{\alpha}z}$, $|\rho| = 1$ for some $\rho \in \partial U, \alpha \in U$

where the Cowen auxiliary functions *g*, σ and *h* of are φ

$\sigma(\ldots) = \frac{\alpha-\bar{\rho}z}{1-\overline{\rho\alpha}z} = \varphi^{-1}(z),\ \ g(z) = \frac{1}{1-\overline{\rho\alpha}z}$ and $h(z) = 1-\bar{\alpha}z.$

$C^*_\varphi = T_g C_\sigma T^*_h$

$W^n_{f,\varphi}W^*_{f,\varphi}$
$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}C^*_\varphi T^*_f$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}T_g C_\sigma\ T^*_h T^*_f$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}T_g C_\sigma\ T_{\bar{h}}T_{\bar{f}}$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.T_{go\varphi_n}C_{\varphi_n}C_\sigma\ T_{\overline{hf}}$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.T_{go\varphi_n}C_{\sigma o\varphi_n}T_{\ldots\bar{f}}$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.T_{go\varphi_n}T_{\ldots\bar{f}o\sigma o\varphi_n}C_{\sigma o\varphi_n}$

$= T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.T_{(go\varphi_n)(\overline{hf}\ldots o\varphi_n)}\ \ C_{\sigma o\varphi_n}$

$= f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1}).(go\varphi_n).\ldots\bar{f}o\sigma o\varphi_n).C_{\sigma o\varphi_n}$

$W^*_{f,\varphi}W^n_{f,\varphi}$
$= C^*_\varphi T^*_f T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}$

$= T_g C_\sigma\ T_{\bar{h}}T_{\bar{f}}T_{f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}$

$= T_g C_\sigma\ T_{\ldots\bar{f}f\ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)\ldots(f\ \ o\varphi_{n-1})}.C_{\varphi_n}$

$= T_g T_{(\bar{h} f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\sigma}.C_\sigma\ C_{\varphi_n}$

$= T_{g\ (\bar{h} f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\sigma]}.C_{\varphi_n o\sigma}$

$= g.(\overline{\ }).\ (f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\sigma.C_{\varphi_n o\sigma}$

We have

$g o\varphi_n(z) = g(\varphi_n(z)) = \frac{1}{1-\bar{\rho}\bar{\alpha}\varphi_n(z)} = \frac{1}{K_\beta\ \ [(1-\overline{\rho\alpha}\omega\beta)-(\overline{\beta}-\overline{\rho\alpha}\omega)z]}$,

$\sigma o\varphi_n(z) = \sigma(\varphi_n(z)) = \frac{\alpha-\varphi_n(z)}{1-\bar{\alpha}\varphi_n(z)} = \frac{(\alpha-\bar{c}\omega\beta)-(\overline{\beta}-\omega)z}{(1-\bar{\alpha}\omega\beta)-(\overline{\beta}-\bar{\alpha}\omega)z}$,

$\overline{h f o\sigma o\varphi_n(z)} = \overline{h f}(\sigma o\varphi_n(z)) = \frac{\overline{f o\sigma o\varphi_n(z)}}{K_\alpha\ \ (\sigma o\varphi_n(z))}$,

$= \overline{\ }(1-\bar{\alpha}[\frac{(\alpha-\bar{c}\omega\beta)-(\overline{\beta}-\omega)z}{(1-\bar{\alpha}\omega\beta)-(\overline{\beta}-\bar{\alpha}\omega)z}])$

$=$

$\overline{f(\sigma\ )}\ (\frac{1-|\alpha|^2-[(1-\bar{\alpha})\overline{\beta}-(\bar{\alpha}-\rho\bar{\alpha})\omega]z}{(1-\overline{\rho\alpha}\omega\beta)-(\overline{\beta}-\bar{\alpha}\omega)z}])$

$g.(\overline{h f o\sigma)(z)} = \frac{\overline{f o\sigma(z)}}{\|K_\alpha\ \ \|^2(1-\bar{\rho}\ z)\overline{(1-\bar{\rho}\ z)}}$

$(\ (f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\sigma(z) = \frac{(1-\overline{\rho\alpha}z)}{\overline{\ }\ )[(1-\alpha\overline{\beta})-\bar{\rho}(\bar{\alpha}-\overline{\beta})z]}$

Hence

$W_{f,\varphi} = \frac{\overline{f(\sigma o\varphi_n)}}{\overline{f(\beta)}}[\frac{(1-|\alpha|^2)-\beta(1-\bar{c})z}{[(1-\bar{\alpha}\omega\beta)-(\overline{\beta}-\bar{\alpha}\omega)z]^2}]\ C_{\sigma o\varphi_n} = I$

$W^n_{f,\varphi} = \frac{\overline{\ }\ )}{\overline{\ }\ )}[\frac{(1-|\alpha|^2)}{(1-\bar{\alpha}\rho z)[(1-\bar{\beta})-\bar{\rho}(\bar{\alpha}-\overline{\beta})z]}]\ C_{\varphi_n o\sigma} = I$

Then $W_{f,\varphi}$ is n-unitary if and only if

$\overline{\ \ \varphi_n)(z)}[\frac{(1-|\alpha|^2)-\beta(1-\bar{\alpha})z}{[(1-\overline{\rho\alpha}\omega\ )-(\overline{\beta}-\overline{\rho\alpha}\omega)z]^2}] =$

$\overline{\ \ )}\ [\frac{(1-|\alpha|^2)}{(1-\bar{\alpha}\rho z)[(1-\bar{\beta})-\bar{\rho}(\bar{\alpha}-\overline{\beta})z]}]$

Proposition: (3-11)

Suppose $\varphi(t) = t\ \ \ for\ some\ \ t \in U$ and f is a linear fractional if $W_{f,\varphi}$ is n-normal, then

$$f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1}) = \frac{(f(t))^n K_t}{K_t(\varphi_n)}$$

Proof:

Suppose that $W_{f,\varphi}$ is n-normal then $W^n_{f,\varphi}$ is normal

$(W^n_{f,\varphi})^* K_t = (W^*_{f,\varphi}(K_t))^n = \overline{f(t)})^n K_t$

Hence K_t is an eigenvector of $(W^n_{f,\varphi})^*$ with corresponding eigenvalue $(\overline{f(t)})^n$, since $W^n_{f,\varphi}$ is normal and by theorem (2-6) we have K_t is an eigenvector for $(W^n_{f,\varphi})$ with corresponding eigenvalue $(f(t))^n$

$W^n_{f,\varphi}\ \ K_t = (f(t))^n K_t$

$T_{f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1})}.C_{\varphi_n} K_t = (f(t))^n K_t$

$f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1}).C_{\varphi_n} K_t = (f(t))^n K_t$

$f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1}).\ \ K_t(\varphi_n) = (f(t))^n K_t$ then we have

$$f\ \ .(f\ \ o\varphi\ \ ).(f\ \ o\varphi_2)....(f\ \ o\varphi_{n-1}) = \frac{(f(t))^n K_t}{K_t(\varphi_n)}$$

Corollary : (3-12)

Suppose $\varphi(0) = 0$ then $W_{f,\varphi}$ is n-normal if and only if f is constant and C_φ is n-normal.

Notation (3-13) Let We denote to the form $f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1})$ by ξ

Proposition: (3-14)

Suppose $\varphi(t)=t$ $for\ some\ t\in U$ then $W_{f,\varphi}$ is n-normal if and only if $\varphi_n=\alpha_t o(\gamma\alpha_t)$ and $\xi=\frac{(f(t))^n K_t}{K_t(\varphi_n)}$ where $\alpha_t=\frac{t-z}{1-\bar{t}z}$

Proof :

Suppose that $W_{f,\varphi}$ is n-normal and let $\xi_t=\frac{1}{f(t)}$,since α_t is an automorphism then by proposition (3-10) W_{ξ_t,α_t} is n-unitary and by theorem (2-10) the operator $P=(W^*{}_{\xi_t,\alpha_t})(W^n{}_{f,\varphi})(W_{\xi_t,\alpha_t})$

is n-normal .

Let the Cowen auxiliary functions g,σ and h of α_t are

$\sigma(z)=\frac{t-z}{1-\bar{t}z}=\alpha_t{}^{-1}(z)=\alpha_t(z),\quad g(z)=\frac{1}{1-\bar{t}z}\quad and\quad h(z)=1-\bar{t}z.$

$$P=(W^*{}_{\xi_t,\alpha_t})(W^n{}_{f,\varphi})(W_{\xi_t,\alpha_t})$$

$$=C^*_{\alpha_t}T^*_{\xi_t}T_{f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1})}.C_{\varphi_n}T_{\xi_t}C_{\alpha_t}$$

$$=T_g\ C_{\alpha_t}T^*_h\ T^*_{\xi_t}T_{f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1})}.C_{\varphi_n}T_{\xi_t}C_{\alpha_t}$$

$$=T_g\ C_{\alpha_t}T_{\frac{1}{f(t)}}T_{f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1})}.C_{\varphi_n}T_{\xi_t}C_{\alpha_t}$$

$$=T_{g(\frac{1}{f(t)}).(f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\alpha_t).(\xi_t o\varphi_n o\alpha_t)}\ C_{\alpha_t o\varphi_n o\alpha_t}$$

Let

$j=g.(\frac{1}{f(t)}).(f\ .(f\ o\varphi\ ).(f\ o\varphi_2)....(f\ o\varphi_{n-1}))o\alpha_t).(\xi_t o\varphi_n o\alpha_t)$

Since $\alpha_t o\varphi_n o\alpha_t(0)=0$ and $P=W_{j,\alpha_t o\varphi_n o\alpha_t}$ then by corollary (3-12) j is a constant map and the composition operator $C_{\alpha_t o\varphi_n o\alpha_t}$ is n-normal and by proposition (3-2)

$\alpha_t o\varphi_n o\alpha_t=\gamma z,\ |\gamma|\le 1$ so $\varphi_n=\alpha_t o(\gamma\alpha_t)$. we can see that $j=\xi(t)$.

Conversely suppose $\varphi_n=\alpha_t o(\gamma\alpha_t)$ and $\xi=\frac{(f(t))^n K_t}{K_t(\varphi_n)}$

$$\varphi_n(z)=\alpha_t(\gamma\alpha_t(z))=\frac{t(1-\gamma)-(|p|^2-\gamma)z}{1-|p|^2\gamma-\bar{t}(1-\gamma)z}$$

and $\xi=\frac{(f(t))^n K_t}{K_t(\varphi_n)}=\frac{1-|p|^2}{1-|p|^2\gamma-\bar{t}(1-\gamma)z}$

Since $C_{\gamma z}$ is n-normal and W_{ξ_t,α_t} is n-unitary , we have

$$W_{\xi_t,\alpha_t}C_{\gamma z}W^*{}_{\xi_t,\alpha_t}=T_{\xi_t}C_{\alpha_t}C_{\gamma z}C^*_{\alpha_t}T^*_{\xi_t}$$

$$=T_{\xi_t}C_{\alpha_t}C_{\gamma z}T_gC_{\alpha_t}T^*_hT^*_{\xi_t}$$

$$=T_{\xi_t}C_{\alpha_t}C_{\gamma z}T_gC_{\alpha_t}\frac{1}{f(t)}\cdot\frac{K_t}{\|K_t\|}$$

$$=\frac{1}{\|K_t\|}.T_{\xi_t}T_{go(\gamma zo\alpha_t)}\ C_{(\alpha_t o\gamma zo\alpha_t)}.$$

$$=\frac{K_t}{\|K_t\|^2}.go(\gamma\alpha_t)C_{(\alpha_t o(\gamma\alpha_t))}$$

If we substituting $g(\gamma\alpha_t(z))=\frac{1-\bar{p}z}{1-|p|^2\gamma-\bar{t}(1-\gamma)z}$, $\varphi_n(z)=\alpha_t(\gamma\alpha_t(z))=\frac{t(1-\gamma)-(|p|^2-\gamma)z}{1-|p|^2\gamma-\bar{t}(1-\gamma)z}$ and

$\xi=\frac{(f(t))^n K_t}{K_t(\varphi_n)}=\frac{1-|p|^2}{1-|p|^2\gamma-\bar{t}(1-\gamma)z}$ in the last formula we obtain that

$W_{\xi_t,\alpha_t}C_{\gamma z}W^*{}_{\xi_t,\alpha_t}$ is similar to $W_{f,\varphi}$ then by proposition (2-10) $W_{f,\varphi}$ is n-normal .

Proposition: (3-15)

Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self map and $f(z) = K_{\sigma(0)}(z) = \frac{d}{cz+d}$ then $W_{f,\varphi}$ is n-normal if and only if $\frac{1}{-\bar{b}z+d}(\xi o\sigma).C_{\varphi_n o\sigma} = \frac{r+sz}{(rd-\bar{b}m)+(sd-\bar{b}r)z}.\xi.C_{\varphi_n o\sigma}$

Proof :

Suppose the Cowen auxiliary functions g, σ and h of are φ then

$$W_{f,\varphi}^{n} W_{f,\varphi}^{*} = f\ .(f\ o\varphi\ ).(f\ o\varphi_2).\ldots(f\ o\varphi_{n-1}).(go\varphi_n).[illegible]fo\sigma o\varphi_n).C_{\sigma o\varphi_n}$$

And

$$W_{f,\varphi}^{*} W[illegible] = [illegible].\ (f\ .(f\ o\varphi[illegible](f\ o\varphi_2).\ldots(f\ o\varphi_{n-1}))o\sigma.C_{\varphi_n o\sigma}$$

$$\overline{(hf o\sigma o\varphi_n)}(z) = \overline{(hf o\sigma)}(\varphi_n(z)) = \bar{h}(\sigma(\varphi_n(z))).\bar{f}(\sigma(\varphi_n(z))) = \bar{d}$$

$$([illegible]) = \frac{r[illegible]}{[illegible] - \bar{b}m[illegible] - \bar{b}r)z}$$

$$(hf[illegible] = \overline{(ho\sigma)}(z).\overline{(fo[illegible])} = \bar{d}$$

$g.\overline{(hfo\sigma)}(z). = \frac{\bar{d}}{-\bar{b}z+d}$ then we have

$$\frac{1}{-\bar{b}z+d}(\xi o\sigma).C_{\varphi_n o\sigma} = \frac{r+sz}{(rd-\bar{b}m)+(sd-\bar{b}r)z}.\xi.C_{\varphi_n o\sigma}$$

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***المؤثر التركيبي الموزون السوي-n والمؤثر التركيبي الموزون الوحدوي -n على فضاء هاردي**

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الخلاصة

في هذا البحث قمنا بتعريف على فضاء هاردي -nالمؤثر التركيبي الموزون الوحدوي-n المؤثر التركيبي الموزون السوي وبرهان بعض الخواص والمبرهنات الخاصة بهما

الكلمات المفتاحية : المؤثر التركيبي .المؤثر التركيبي الموزون.المؤثر السوي .فضاء هاردي

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