Collocation Orthonormal Bernstein Polynomials Method for Solving Integral Equations

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Received on:14/1/2015 & Accepted on:17/9/2015

ABSTRACT:

In this paper, we use a combination of Orthonormal Bernstein functions on the interval $[0,1]$ for degree $m = 5$, and 6 to produce a new approach implementing Bernstein operational matrix of derivative as a method for the numerical solution of linear Fredholm integral equations and Volterra integral equations of the second kind. The method converges rapidly to the exact solution and gives very accurate results even by low value of m. Illustrative examples are included to demonstrate the validity and efficiency of the technique and convergence of the method to the exact solution.

Keywords: Bernstein polynomials, Operational Matrix of Derivative, Linear Fredholm Integral Equations of the Second Kind and Volterra Integral Equations.

طريقة العد لمتعددة حدود برنشتن المتعامدة لحل المعادلات التكاملية

الخلاصة:

في هذا البحث تم استخدام مجموعة متعددات لبرنشتن المتعامدة ضمن الفترة [0,1]للدرجة 5,6 وايجاد مصفوفة جديدة للمشتقات واستخدامها لحل المعادلات التكاملية الخطية بنوعيها فردهولم وفولتيرا من النوع الثاني لتعطي نتائج دقيقة للغاية حتى عند انخفاض قيمة m, الامثلة التوضيحية دليل على صحة وكفاءة هذه التقنية والتقارب للحل المضبوط.

INTRODUCTION:

In the Survey of solutions of integral equations, a number of analytical but a few approximate methods for solving numerically various classes of integra equations [1]. Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamical systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations. While in recent years, the interest is in the solution of integral and differential

https://doi.org/10.30684/etj.2015.117189
2412-0758/University of Technology-Iraq, Baghdad, Iraq

equations, such as Fredholm, Volterra, and integro-differential equations. The general form of Fredholm, Volterra integral equations of the second kind respectively are

1- Freholm integral equation: (FIE)

$$u(x) = g(x) + \int_a^b k(x,t)u(t)dt \qquad x \in [a,b] \qquad \dots (1)$$

2- Volterra integral equation: (VIE)

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \qquad x \in [0,1] \qquad \dots (2)$$

Integral equations are widely used for solving many problems in mathematical physics and engineering. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions [2, 3], Hybrid Legendre and Block-Pulse functions. Bernstein polynomials play a prominent role in various areas of mathematics. These polynomials have been frequently used in the solution of integral equations, differential equations and approximation theory, see [4]. Recently the various operational matrices of the polynomials have been developed to cover the numerical solution of differential, integral and integro-differential equations. In [5] the operational matrices of Bernstein polynomials are introduced. Yousefi et al.in [6], [7] and [8] have presented Legendre wavelets and Berntein operational matrices and used them to solve miscellaneous systems. Another motivation is concerned with the direct solution techniques for solving the Fredholm and Volterra integral equations respectively on the interval [0,1] using the method based on the derivatives of orthonormal (B-polynomials) sense for m=5 and 6. Finally, the accuracy of the proposed algorithm is demonstrated by test problems.

Bernstein Polynomials (B-Polynomials):

The Bernstein Polynomials (B-Polynomials) [9],[13], are some useful polynomials defined on [0,1]. The Bernstein Polynomials of degree m form a basis for the power polynomials of degree m. we can mentioned, B-Polynomials are aset of Polynomials

$$B_{k,m}(x) = \binom{m}{k} x^k (1-x)^{m-k} \quad k = 0,1,\dots,m \qquad \dots (3)$$

Where

$\binom{m}{k} = \frac{m!}{k!(m-k)!}$ and $B_{k,m} = 0$ for $k < m$ or $k > m$

Note that each of these m+1 Polynomials having degree m and they form a partition of unitty, i.e

$\sum_{k=0}^{m} B_{k,m}(x) = 1$, also $B_{k,m}(x)$ in which $k \notin \{0, m\}$ has asingle unique local maximum of $k^k m^{-m}(m-k)^{m-k}\binom{m}{k}$, it can provide flexibility applicable to impose boundary conditions at the end points of the interval. First derivative of the generalized Bernstein basis polynomials.

$$\frac{d}{dx} B_{k,m}(x) = m\left[B_{k-1,m-1}(x) + B_{k,m-1}(x)\right] \qquad \dots (4)$$

In this paper, we use $\Psi_m(x)$ notation to show

$$\Psi_m(x) = [B_{0m}(x),\ B_{1m}(x), \dots, B_{mm}(x)]^T \qquad \dots (5)$$

Where we can have

$$\Psi_m(x) = A_m(x)\Delta_m(x) \qquad \dots (6)$$

That A is the matrix and $(k+1)^{th}$ row of A is

$$A_{k+1} = [0,0,\dots^{k\ times}, 0, s\_(0,k,m), s\_(1,k,m), \dots, s\_(m,k,m)\ ]$$

$$=\left[0,0,\dots^{k\ times}, 0, (-1)^0\binom{m}{k}\binom{m-k}{0}, (-1)^1\binom{m}{k}\binom{m-k}{1}, \dots, (-1)^{m-k}\binom{m}{k}\binom{m-k}{m-k}\right] \dots (7)$$

Where

$$S_{i,k,m} = (-1)^i\binom{m}{k}\binom{m-k}{i}$$

And $\Delta_n(x) = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^m \end{bmatrix}$ ...(8)

Using MATHEMATICA code, the six (B-Polynomials) of degree five over the interval[0,1], are given

$B_{05}(x) = (1-x)^5$
$B_{15}(x) = 5x(1-x)^4$
$B_{25}(x) = 10x^2(1-x)^3$
$B_{35}(x) = 10x^3(1-x)^2$
$B_{45}(x) = 5x^4(1-x)$
$B_{55}(x) = x^5$

And the seven (B-Polynomials) of degree six over [0,1] are given

$B_{06}(x) = (1-x)^6$
$B_{16}(x) = 6x(1-x)^5$
$B_{26}(x) = 15x^2(1-x)^4$
$B_{36}(x) = 20x^3(1-x)^3$
$B_{46}(x) = 15x^4(1-x)^2$
$B_{56}(x) = 6x^5(1-x)$
$B_{66}(x) = x^6$

(B-Polynomials) Approximation:

Let u(x) is a continuous function on [0,1]. Then u(x) can be written as $u(x) = \lim_{m\to\infty}\sum_{k=0}^{m} u\left(\frac{k}{m}\right) B_{km}(x)$ uniformly on the interval [0,1] .

$$u(x) \approx \sum_{k=0}^{m} c_k B_{km}(x) = C^T \varnothing(x) \qquad \dots. (9)$$

Where

$\varnothing^T(x) = [B_{0m}(x), B_{1m}(x), \dots, B_{mm}(x)]$ and

$C^T = [c_0, c_1, \dots, c_m]$ can be calculated by:

$$C^T = \left(\int_0^1 u(x)\emptyset^T(x)dx\right) Q^{-1}, \qquad \dots (10)$$

Where

Q is an $(m+1)\times(m+1)$ matrix and is said dual matrix of $\varnothing(x)$

$$Q(x) = (\varnothing(x), \varnothing(x)) = \int_0^1 \emptyset(x)\,\emptyset^T(x)dx$$

$$=\int_0^1 \left(A\Delta_n(x)\right)\left(A\Delta_n(x)\right)^T dx$$
$$= A\left[\int_0^1 \Delta_n(x)\Delta_n^T(x)dx\right]A^T$$
$$= AHA^T, \qquad \ldots(11)$$

A is defined by eqs.(7) and H is a Hilbert matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{2m+1} \end{bmatrix} \text{ and}$$

$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_m \end{bmatrix}$$

The elements of the dual matrix Q, are given explicity by

$$(Q_m)_{k+1,i+1} = \int_0^1 B_{km}(x)B_{im}(x)dx$$
$$= \binom{m}{k}\binom{m}{i}\int_0^1 (1-x)^{2m-(k+i)}x^{k+i}dx \qquad \ldots(12)$$

where

$k, i = 0,1,\ldots,m$

The Derivative for Orthonormal (B-Polynomials):

The representation of the orthonormal Bernstein Polynomials, denoted by $b_{i5}(x), b_{i6}(x)$ here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of degree five and six.

Then the following sets of orthonormal polynomials [10], [11].

$$b_{05}(x) = \sqrt{11}(1-x)^5$$
$$b_{15}(x) = 6\left[5x(1-x)^4 - \frac{1}{2}(1-x)^5\right]$$
$$b_{25}(x) = \frac{18\sqrt{7}}{5}\left[10(1-x)^3x^2 - 5(1-x)^4x + \frac{5}{18}(1-x)^5\right]$$
$$b_{35}(x) = \frac{28}{\sqrt{5}}\left[10(1-x)^2x^3 - 15(1-x)^3x^2 + \frac{30}{7}(1-x)^4x - \frac{5}{28}(1-x)^5\right]$$
$$b_{45}(x) = 7\sqrt{3}\left[5(1-x)x^4 - 20(1-x)^2x^3 + 18(1-x)^3x^2 - 4(1-x)^4x + \frac{1}{7}(1-x)^5\right]$$
$$b_{55}(x) = 6\left[x^5 - \frac{25}{5}(1-x)x^4 + \frac{100}{3}(1-x)^2x^3 - 25(1-x)^3x^2 + 5(1-x)^4x - \frac{1}{6}(1-x)^5\right]$$

and

$$b_{06}(x) = \sqrt{13}(1-x)^6$$
$$b_{16}(x) = \sqrt{44}\left[6x(1-x)^5 - \frac{1}{2}(1-x)^6\right]$$

$b_{26}(x) = 11\left[15(1-x)^4x^2 - 6(1-x)^5x + \frac{3}{11}(1-x)^6\right]$
$b_{36}(x) = \sqrt{252}\left[20(1-x)^3x^3 - \frac{45}{2}(1-x)^4x^2 + 5(1-x)^5x - \frac{11}{66}(1-x)^6\right]$
$b_{46}(x) = \frac{42}{\sqrt{5}}\left[15(1-x)^2x^4 - 40(1-x)^3x^3 + \frac{180}{7}(1-x)^4x^2 - \frac{30}{7}(1-x)^5x + \frac{5}{42}(1-x)^6\right]$
$b_{56}(x) = \frac{28}{\sqrt{3}}\left[6x^5(1-x) - \frac{75}{2}(1-x)^2x^4 + 60(1-x)^3x^3 - 30(1-x)^4x^2 + \frac{30}{7}(1-x)^5x - \frac{3}{28}(1-x)^6\right]$
$b_{66}(x) = 7\left[x^6 - 18(1-x)x^5 + 75(1-x)^2x^4 - 100(1-x)^3x^3 + 45(1-x)^4x^2 - 6t(1-x)^5 + \frac{1}{7}(1-x)^6\right]$

In addition, the explicit representation for the orthonormal Bernstein polynomials as [12]

$$b_{km}(x) = \left(\sqrt{2(m-k)+1}\right)(1-x)^{m-k}\sum_{i=0}^{k}(-1)^i\binom{2m+1-i}{k-i}\binom{k}{i}x^{k-i} \quad \dots (13)$$

The eqs(13) can be written in terms of the Bernstein basis functions as

$$b_{km}(x) = \left(\sqrt{2(m-k)+1}\right)\sum_{i=0}^{k}(-1)^i\frac{\binom{2m+1-i}{k-i}\binom{k}{i}}{\binom{m-i}{k-i}}B_{k-i,m-i}(x) \quad \dots (14)$$

Any generalized Bernstein basis polynomials of degree m can be written as a linear combination of the generalized Bernstein basis polynomials of degree m+1

$$B_{k,m}(x) = \frac{m-k+1}{m+1}B_{k,m+1}(x) + \frac{k+1}{m+1}B_{k+1,m+1}(x) \quad \dots(15)$$

By utilizing eqs (15),the following functions can be written as

$$B_{k,m-1}(x) = \frac{m-k}{m}B_{k,m}(x) + \frac{k+1}{m}B_{k+1,m}(x) \quad \dots (16)$$

and

$$B_{k-1,m-1}(x) = \frac{m-k+1}{m}B_{k-1,m}(x) + \frac{k}{m}B_{k,m}(x) \quad \dots(17)$$

Substituting these eqs (16) and (17) into the right hand side of the eqs(4), we get the following derivatives of Bernstein basis polynomials

$$\frac{d}{dx}B_{k,m}(x) = (m-k+1)B_{k-1,m}(x) + (2k-m)B_{k,m}(x) - (k+1)B_{k+1,m}(x) \quad \dots(18)$$

In [10], the derivative of the orthonormal (B-Polynomials) of degree five are introduced as given

$b'_{05} = \left[-5\sqrt{11}\,B_{05} - \sqrt{11}B_{15}\right]$
$b'_{15} = [45B_{05} - 15B_{15} - 2B_{25}]$
$b'_{25} = \left[23\sqrt{7}\,B_{05} + \frac{121\sqrt{7}}{5}B_{15} + \frac{18\sqrt{7}}{5}B_{25} - \frac{54\sqrt{7}}{5}B_{35}\right]$
$b'_{35} = \left[\frac{145}{\sqrt{5}}B_{05} - \frac{235}{\sqrt{5}}B_{15} + \frac{78}{\sqrt{5}}B_{25} + \frac{154}{\sqrt{5}}B_{35} - \frac{113}{\sqrt{5}}B_{45}\right]$
$b'_{45} = \left[-33\sqrt{3}B_{05} + \frac{331\sqrt{3}}{5}B_{15} - \frac{217}{5}B_{25}\frac{126\sqrt{3}}{5}B_{35} + 77\sqrt{3}B_{45} - 35\sqrt{3}B_{55}\right]$
$b'_{55} = [35B_{05} - 55B_{15} + 63B_{25} + 35B_{35} - 119B_{45} + 105B_{55}]$

And in [11] the derivative of the orthonormal (B-Polynomials) for degree six are introduced as given

$b'_{06} = \left[-6\sqrt{11}\, B_{06} - \sqrt{13}B_{16}\right]$
$b'_{16} = \left[8\sqrt{11}B_{06} - 7\sqrt{11}B_{16} - 4\sqrt{11}B_{26}\right]$
$b'_{26} = \left[-84\, B_{06} + 96B_{16} + B_{26} - 3B_{36}\right]$
$b'_{36} = \left[36\sqrt{7}B_{06} - 64\sqrt{7}B_{16} + 32\sqrt{7}B_{26} + 27\sqrt{7}B_{36} - 24\sqrt{7}B_{46}\right]$
$b'_{46} =$
$\left[-42\sqrt{5}B_{06} + 95\sqrt{5}B_{16} - 150.263681B_{26} - 40.24922359B_{36} + 187.8297101B_{46} - 42\sqrt{5}B_{56}\right]$
$b'_{56} =$
$\left[46\sqrt{3}B_{06} - 206.1140461B_{16} + 235.5589098B_{26} - 14\sqrt{3}B_{36} - 242.4871131B_{46} + 266.7358244B_{56} - 56\sqrt{3}B_{66}\right]$
$b'_{66} = \left[-48B_{06} + 132B_{16} - 168B_{26} + 42B_{36} + 168B_{46} - 252B_{56} + 168B_{66}\right]$

Second kind integral equations:

In this section, we use Orthonormal Polynomials for solving second kind Fredholm and Volterra integral equations.

1- Fredholm integral equation (FIE):

Where in eq. (1) $g(x) \in L^2[0,1], k(x,t) \in L^2([0,1] \times [0,1])$ are known and $u(t)$ is unknown function to be determined.

First we assume the unknown functions

$$u_i(x) = C_i^T B(x), i = 1,2,\dots,n \qquad \dots(19)$$

by substituting (19)in (1) we have:

$$C_i^T B(x) = g_i(x) + \int_0^1 k_{i,j}(x,t)\, C_i^T B(t)dt$$

$$C_i^T B(x) - \int_0^1 k_{i,j}(x,t)\, C_i^T B(t)dt = g_i(x) \qquad \dots(20)$$

Pick distinct node points $t_1, t_2, \dots, t_n \in [0,1]$

This leads to determining $\{ c_1, c_2, \dots, c_n\}$ as the solution of linear system

$$\sum_{i=1}^{n} C_j \left[B(x_i) - \int_0^1 k(x_j,t_i)\, B(t_i)dt\right] = g(x_i) \qquad \dots(21)$$

In this paper Collocation points are $t_i = \frac{i}{n}\ for\ i = 1,2,\dots,n$ so that we have a system of linear equations

$$L_n X = l_n \quad \text{where} \quad L_n = \left|B(x_i) - \int_0^1 k(x_j,t_i)\, B(t_i)dt\right|_{i=0}^{n} \qquad j = 1,2,\dots,n$$

$$l_n = [g(x_i)], \qquad i = o,1,\dots,n$$

2- Volterra integral equation (VIE):

Similarly above section by using Collocation points $t_i = \frac{i}{n}\ for\ i = 1,2,\dots,n$

$$L_n = \left|B(x_i) - \int_0^x k(x_j,t_i)\, B(t_i)dt\right|_{i=0}^{n} \qquad j = 1,2,\dots,n$$

$$l_n = [f(x_i)], \qquad i = o,1,\dots,n$$

Numerical Results:

In this section VIE, FIE are considered and solved by the introduced method.

Example 1: Consider the following FIE

$$u(x) = \sin x + \int_0^1 (1 - x\cos xt)\, u(t)dt \qquad \dots.(22)$$

The exact solution u(x)=1. Solving eqs. (20) and (21) we get the values of C^{T}

C= $[0.91402903 \quad 0.10091438 \quad 3.48867166 \quad -2.7619172 \quad 3.51373945 \quad 0.15379915 \quad 0.98896685]^T$

Table1 shows the numerical results for this example.

Table (1):some numerical results for example 1

x	Approximat solution $b_n(x)$	Approximat solution $B_n(x)$	$AbsouteError$ $\|exact - B_{n6}\|$
0	0.91402333	0.99993256	6.7440e-005
0.1	0.93422123	0.99993267	6.7330e-005
0.2	0.96878356	0.99994532	5.4680e-005
0.3	0.96878320	0.99994444	5.5560e-005
0.4	0.97843933	0.99995324	4.6760e-005
0.5	0.93270466	0.99995417	4.5830e-005
0.6	0.99388042	0.99996618	3.3820e-005
0.7	0.99963715	0.99997654	2.3460e-005
0.8	0.98888323	0.99998790	1.2100e-005
0.9	0.99999668	0.99999999	1.0000e-008
1	0.99999998	1.00000000	0.00000000
			$M.S.E$ =6.7440e-005 $L.S.E$=2.1286e-008

Example 2: Consider the following FIE

$$u(x) = e^{-x} - \int_0^1 xe^t u(t)dt$$

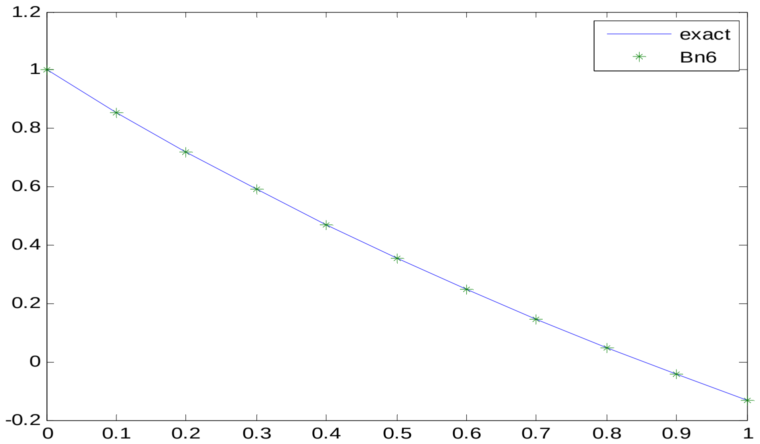
the exact solution $u(x) = e^{-x} - \frac{x}{2}$. Solving eqs(20) and (21) we get the values of C^{T}

C=$[1 \quad 1.13944655 \quad -0.05603088 \quad 0.85196593 \quad -0.13491534 \quad 0.11655846 \quad -0.16383600]^T$

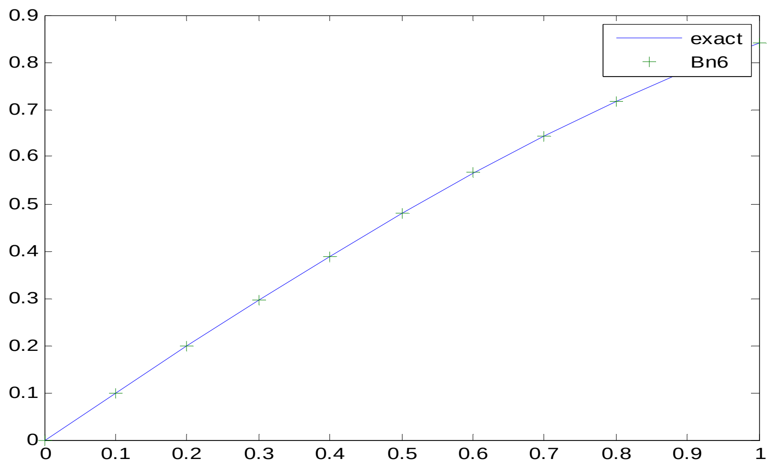
Table2 shows the numerical results for this example.

Table(2): some numerical results for example 2

x	Exact solution	Approximate solution $b_n(x)$	Approximate solution $B_n(x)$	$AbsouteError$ $\|exact - B_{n6}\|$
0	1	1	1	0.00000000
0.1	0.85483742	0.94188967	0.85488967	0.00005225
0.2	0.71873075	0.76843118	0.71811760	0.00008101
0.3	0.59081822	0.59503874	0.59081874	0.00000052
0.4	0.47032005	0.46240281	0.47032115	0.00000110
0.5	0.35653066	0.35230187	0.35653066	0.00352200
0.6	0.24881164	0.24605093	0.24880053	0.00001111
0.7	0.14658530	0.13907957	0.14658510	0.00000020
0.8	0.04932896	0.04047384	0.04932895	0.00000001
0.9	-0.04343034	-0.04663478	-0.04343155	0.00000121
1	-0.13212056	-0.16383600	-0.13212056	0.00000000
	$M.S.E$ =0.00352200 $L.S.E$=0.00000000			



Figure(1) of example1



Figure(2) of example3

Example 3: Consider the following VIE

$u(x) = x - \int_0^x (x-t)u(t)dt$ ...(23)

The exact solution $u(x) = \sin x$. Table (3) shows the numerical results for this example(3)

$C=$

$[0 \quad 0.15606151 \quad 0.37680718 \quad 0.40682330 \quad 0.73751386 \quad 0.68080388 \quad 0.84172197]^T$

Table(3):some numerical results for example 3

x	Exact solution	Approximat solution $b_n(x)$	Approximat solution $B_n(x)$	$AbsouteError$ $\lvert exact - B_{n6}\rvert$
0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.09983342	0.09924030	0.09983389	0.00000059
0.2	0.19866933	0.19972478	0.19865878	0.00001055
0.3	0.29552021	0.29617066	0.29552053	0.00000065
0.4	0.38941834	0.38930420	0.38941820	0.00000014
0.5	0.47942554	0.47990931	0.47942560	0.00000048
0.6	0.56464247	0.56604342	0.56604342	0.00140095
0.7	0.64421769	0.64342047	0.64342047	0.00007972
0.8	0.71735609	0.70896119	0.71732785	0.00008394
0.9	0.78332691	0.76751046	0.78331110	0.00001581
1	0.84147098	0.84172197	0.84147073	0.00000025
	$M.S.E$ =0.00140095 $L.S.E$=0.00000000			

CONCLUSION:

In this work, VIE, FIE have been solved by using Bernstein basis polynomials of degree m in collocation method. Comparison of the approximate solutions and the exact solutions show that the proposed method is efficient tool. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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