WHITNEY THEOREM FOR COPOSITIVE APPROXIMATION IN $L_{\psi,p}(I), \quad p<1$

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Abstract : In this research, we have important results about finding the relationship between the best approximation degree and the so called $\tau-$modulus (or Sendov- Popov modulus) of order k in the space $L_{\psi,p}(I), \quad p<1$, and the polynomial is copositive with the function f at the points in an interval $I=[-b,b]$, and also found assessment between the best approximation degree by algebraic polynomial of degree $\le k-1$, and the modulus of smoothness of degree $\le k$, to the function $f \in L_{\psi,p}(I) \cap \Delta^0(J_s)$, $p<1$.

1.1. INTRODUCTION , DIFINITIONS AND MAIN RESULTS

The theory of Whitney is one of the achievements of scientist Hassler Whitney in approximation theory . The following theory called Whitney theorem , which provides the following : (Let $f \in L_p[a,b], 0<p\le\infty$, then there exists $q_{k-1} \in \Pi_{k-1}$,a polynomial of degree $\le k-1$,such that

$$\|f-q_{k-1}\|_{L_p[a,b]} \le c\omega_k(f,b-a,[a,b])_p$$

Whitney theorem was proved by Burkill [6] when ($k=2, p=\infty$), Whitney ([6],[7]) when ($p=\infty$), Brudnyi [12] when ($1\le p<\infty$), Storozhenko [4] when ($0<p<1$). In [9] K.A.kopotun proved the Whitney theorem of type k-monotone functions . In(2003) E.S. Bhaya [5] proved in theorem (2.1.1) the Whitney theorem of interpolators type for k-monotone functions for K.A.Kopotun:

Theorem 1.1.1: Let $m,k \in N, m<k$ and $f \in \Delta^k \cap W_p^m(I)$ .Then for any, $n \ge k-1$, there exists a polynomial $p_n \in \Pi_n$ such that :

$$\left\|f^{(j)}-p_n^{(j)}\right\|_p \le c(p,k)\omega_{k-j}^{\varphi}(f^{(j)},n^{-1},I)_p \quad \text{for } j=1,\dots,m .$$

The classical Whitney theorem establishes the equivalence between the modulus of smoothness $\omega_r(f,|I|,I)_p$ and the error best approximation $E_r(f)_p$ of a function $f: I \to R$ by algebraic polynomials of degree $\le r-1$ in $L_p, 1\le p<\infty$ [5].

1.1.2.THE WEIGHTED QUASI NORMED SPACE

The weighted normed linear space $L_{\psi,p}(I)$, $p<1$, which is the set of all functions f on the interval $I \subset \Re$, $I=[-b,b]$, b is a positive integer and ψ is increasing function called weight , hat is the weighted quasi normed space can be define in form

$$L_{\psi,p}(I)=\{f,f:I\subset\Re\to\Re:\left(\int_I\left|\frac{f(t)}{\psi(t)}\right|^p dt\right)^{\frac{1}{p}}<\infty,\quad p<1\}.$$

And the (quasi) norm $\|f\|_{L_{\psi,p}(I)}<\infty$, where as always,

$$\|f\|_{L_{\psi,p}(I)}=\left(\int_I\left|\frac{f(t)}{\psi(t)}\right|^p dt\right)^{\frac{1}{p}}\ ,\ t\in I \qquad \dots (1.1.3$$

1.1.4.THE SPACE $L_{\psi,p}(I)$, $p<1$

Let f function in $L_{\psi,p}(I)$, $p<1$ quasi-normed spaces , where $I=[-b,b]$, be an interval such that $I\subset\Re$ and the function ψ is a positive , that is $f(t)\psi(t)\geq 0$ for every $f(t)\geq 0$ and $t\in I$.

The different structure of the spaces $L_{\psi,p}$, $0<p<1$ and the numerous questions by others lead us to understand the need for the following few facts about $L_{\psi,p}$, $p<1$.

The study of approximation will be using polynomials, which will represented by the symbol p . The polynomials used in our work differ in the form and according to the degree of what we want to achieve in the proof . Let $s\geq 0$ and let $J_s=\{j_i\}_{i=1}^s$ be the collections of points, so that :

$j_{s+1}=-b<j_s<\ldots<j_1<b=j_0$, where for $s=0$, $J_0=\phi$. We set $p_n(t)=\prod_{i=1}^{s}(t-j_i)$.

and we let $\Delta^0(J_s)$ be the set of functions f which change their sign exactly at the points $j_i\in J_s$, and we will write $f\in\Delta^0$. Note that our assumption is equivalent to $f(t)\Pi(t,J_s)\geq 0$, $-b\leq t\leq b$. By ([11])for $0<q<p$,and by the same method there exists $c(q)<\infty$, such that

$$\|f\|_{L_{\psi,q}(I)}\leq\|f\|_{L_{\psi,p}(I)}\leq c(q)\|f\|_{L_{\psi,q}(I)}.$$

We consider the space $L_{\psi,p}$,consisting of all functions f on an interval I for which

$$\|f\|^p_{L_{\psi,p}(I)}=\int_I\left|\frac{f(t)}{\psi(t)}\right|^p dx<\infty\ .$$

1.1.5. MODULUS OF SMOOTHNESS

The modulus of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation theory. More subtle measurement are provided by the modulus of smoothness. We will use modulus of smoothness which are connected with difference of higher orders.

For every function f we define the *k*th symmetric difference ([10]) of $f\in L_{\psi,p}(I)$, is given by

$$\Delta_h^k(f,t,I)_\psi := \Delta_h^k(f,t)_\psi := \begin{cases} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\dfrac{f(t-\frac{kh}{2}+ih)}{\psi(t+\frac{kh}{2})}, & t\pm\dfrac{kh}{2}\in I \\ 0 & , \ o.w. \end{cases} \quad \text{where}$$

$\binom{k}{i} = \dfrac{k!}{i!(k-i)!}$, is the binomial coefficient.

The kth usual modulus of smoothness ([3]) of a function $f \in L_{\psi,p}(I)$,defined by

$$\omega_k(f,\delta,I)_{\psi,p} := \sup_{0<h\le\delta}\left\|\Delta_h^k(f,.)\right\|_{L_{\psi,p}(I)}, \quad \delta\ge 0 \qquad \ldots (1.1.6$$

$$\omega_k(f,\delta,I)_{\psi,p} = \sup_{0<h\le\delta}\left\|\left\{\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{f(t-\frac{kh}{2}+ih)}{\psi(t+\frac{kh}{2})}\right\}\right\|_{L_{\psi,p}(I)}$$

The so called $\tau-$modulus (or sendov-popov modulus) ([8]) an averaged modulus of smoothness, defined for bounded measurable functions on I by:

$$\tau_k(f,\delta,I)_{\psi,p} = \left\|\omega_k(f,.,\delta)\right\|_{L_{\psi,p}(I)}$$

Where

$$\omega_k(f,t,\delta)_\psi = \sup\left\{\left|\frac{\Delta_h^k(f,y)}{\psi(y+\frac{kh}{2})}\right| : y\pm\frac{kh}{2}\in\left[t-\frac{k\delta}{2},t+\frac{k\delta}{2}\right]\cap[-b,b]\right\}$$

is the kth local modulus of smoothness ([1])of f .From the definition one can easily see

$$\tau_k(f,\delta,I)_{\psi,\infty} = \omega_k(f,\delta,I)_{\psi,\infty}$$

A new way of measuring smoothness was introduced by Ditzian and Totik ([13]). The Ditzian-Totik modulus of smoothness of $f \in L_{\psi,p}(I)$, $p<1$ which is defined for such an f as follows:

$$\omega_k^\varphi(f,\delta,I)_{\psi,p} = \sup_{0<h\le\delta}\left\|\Delta_{h\varphi(.)}^k(f,.)\right\|_{L_{\psi,p}(I)}, \quad I=[-b,b] \qquad \ldots (1.1.7$$

After this introduction , the main results which wants to prove :

Theorem 1.1.8.(Whitney Theorem) Let $f \in L_{\psi,p}(I)\cap\Delta^0(J_s)$, $p<1$, and let $g_{k-1}\in\Pi_{k-1}\cap\Delta^0(J_s)$, $k>1$, interpolate f at $k-1$ points in side J_A where $J_A = \left[-b+\mu|I|b-\mu|I|\right]$, then

$$\left\|f-g_{k-1}(f)\right\|_{L_{\psi,p}(I)} \le C(p,k)\omega_k^\varphi(f,|I|,I)_{\psi,p}$$

Theorem 1.1.9. Let $f \in L_{\psi,p}(I) \cap \Delta^0(J_s)$, $p<1$ then there exist a polynomial $p_{k-1} \in \Pi_{k-1} \cap \Delta^0(J_s)$, $k>1$ satisfy :

$$\|f - p_{k-1}\|_{L_{\psi,p}(I)} \le C(p,k)\tau_k(f,|I|,I)_{\psi,p}$$

1.2. NEW CHEBYSHEV PARTITION

We have used in this paper the following notations , facts also the partition of period ℓ_i , therefore we found new Chebyshev partition , which is take the form:
$X_j = a\cos\frac{j\pi}{n}$, $a =$positive integer such that $1 \le a < \infty$, $0 \le j \le n$, to an interval I . Now we denote $I_j = [X_{j+1}, X_j]$ $h_j = |I_j| = X_j - X_{j+1}$, $0 \le j \le n$, and $\Delta_n(t) = \frac{\varphi(t)}{n} + \frac{1}{n^2}$ hence $c_1\Delta_n(t) \le h_j \le c_2\Delta_n(t)$ for $t \in I_j$.

For $J_s = \{j_1,...,j_s | j_0 = -b < j_1 < ..., j_s < b = j_{s+1}\}$ we denote by $\Delta^0(J_s)$ the set of all functions $f \in L_{\psi,p}(I) \cap \Delta^0(J_s)$ has $0 \le s < \infty$ change sign k times in J_s [2],in particular if $s = 0$, then $\Delta^0 = \Delta^0(J_0)$ denotes the set of all nonnegative functions on $[-b,b]$.

Let $\delta = \min_{0 \le i \le s} |j_{i+1} - j_i|$ where $j_0 = -b$ and $j_{s+1} = b$. If $j_i \in (X_{j(i)+1}, X_{j(i)})$, $i = 1,...,s$ then it is convenient to denote $j_i^{(v)} \le X_{j(i)+1}$ and $j_i^{(k-1)} \ge X_{j(i)}$, $k>1$ such that $j_i' < j_i'' < ... < j_i^{(k-1)}$ that is

$$j_i \in \left(j_i^{(v)}, j_i^{(k-1)}\right), \quad v = 1,...,k-2 \ , \ \ell_i = [j_i^{(v)}, j_i^{(k-1)}] \ , \quad \text{and } J_i = \left[\frac{j_i + j_i^{(v)}}{k-1}, \frac{j_i + j_i^{(k-1)}}{k-1}\right],$$

$1 \le j \le n$, then $c_1 h_j \le |\ell_i| = (k-1)|J_i| \le c_2 h_j$, where c is a positive number , $i = 1,...,s$, and there for , we get the following facts which we used to prove many results

$$|\ell_i| \approx |J_i| \approx h_j \approx \Delta_n(t) \text{ also } n|\ell_i| \approx n\Delta_n(t) \approx \varphi(t) \text{ for } t \in \ell_i \qquad \text{... (1.2.1}$$

We would like to point out that the symbol ℓ_i , $v = 1,...,k-1$, not represent a derivatives but a symbol of a set of points that exist between $X_{j(i)}$ and $X_{j(i)+1}$, meaning within an period ℓ_i , and $I = \bigcup_{i=1}^{s} \ell_i$, we proved many results and theories on the period ℓ_i, and the fact that the periods ℓ_i , $i = 1,...,s$ isomorphic and have the same properties , so is the proof of these results is true on the aggregate period I . In [5] , recall that for any continuous function f on $[a,b]$ there exist an algebraic polynomial p_{k-1} of degree $\le k-1$ interpolating f inside $[a,b]$,such that

$$\|f - p_{k-1}\|_{L_p[a,b]} \le c(p)\omega_k(f, b-a, [a,b])_p \qquad \text{... (1.2.2}$$

1.3.AUXILIARY RESULTS

Our aim in Auxiliary results is to present the following Lemma and demonstrate its , which are important to complete the target which we want to reach it.

Lemma 1.3.1. Let $J_i \subset \ell_i$ and $f \in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$, $p<1$.Then there exist $p_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$, points in side J_i , then for any constant $\mu > 0$, we have two cases:

Case (1): For $\widetilde{a} = \dfrac{j_i + j_i^{(k-1)}}{k-1} + \mu|J_i| < j_i^{(k-1)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\frac{j_i+j_i^{(v)}}{k-1},\widetilde{a}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\frac{j_i+j_i^{(v)}}{k-1},\widetilde{a}]}$$

Case (2): For $\widetilde{b} = \dfrac{j_i + j_i^{(v)}}{k-1} - \mu|J_i| > j_i^{(v)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(k-1)}}{k-1}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(k-1)}}{k-1}]}$$

Proof:

Case (1) :Let $J_i \subset \ell_i$, and suppose $p_{k-1}(f) = \sum_{i=1}^{s} f(j_i) \prod_{\substack{i=1 \\ 0\le j \le n}}^{s} (t_j - j_i)$, be a linear polynomial of degree $\le k-1$, interpolating f inside J_i and belongs to $\Delta^0(\ell_i)$.Since $f(j_i) \ge 0$, $\forall i = 1,...,s$,and we now that $p_{k-1}(f)$ is nondecreasing for $j_i > t_j$, and hence $p_{k-1}(f) \ge 0$ for $j_i > t_j$ (since $f(t_i) \ge 0$).

Thus $f - p_{k-1}(f) \ge 0$, changes sign in side ℓ_i. In particular $f - p_{k-1}(f) \ge 0$ for $j_i^{(k-1)} > \dfrac{j_i + j_i^{(k-1)}}{k-1}$, hence $p_{k-1}(f) \le f$ for $\dfrac{j_i + j_i^{(k-1)}}{k-1} < \dfrac{j_i + j_i^{(k-1)}}{k-1} + \mu|J_i| < j_i^{(k-1)}$,

then for any constant $\mu > 0$ such that :

$\widetilde{a} = \dfrac{j_i + j_i^{(k-1)}}{k-1} + \mu|J_i| < j_i^{(k-1)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\frac{j_i+j_i^{(k-1)}}{k-1},\widetilde{a}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\frac{j_i+j_i^{(k-1)}}{k-1},\widetilde{a}]} .$$

Since $|J_i| \approx \left\| \left[\dfrac{j_i + j_i^{(k-1)}}{k-1}, \widetilde{a} \right] \right\|$, we conclude that

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\frac{j_i+j_i^{(v)}}{k-1},\widetilde{a}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\frac{j_i+j_i^{(v)}}{k-1},\widetilde{a}]} . \qquad \text{… (1.3.2}$$

Case (2): By the same method in case (1) and in particular $f - p_{k-1}(f) \ge 0$

for $j_i^{(v)} < \frac{j_i + j_i^{(v)}}{k-1}$, hence $p_{k-1}(f) \le f$ for $j_i^{(v)} < \frac{j_i + j_i^{(v)}}{k-1} + \mu|J_i| < \frac{j_i + j_i^{(v)}}{k-1}$, then for any constant $\mu > 0$,such that

$\widetilde{b} = \frac{j_i + j_i^{(v)}}{k-1} - \mu|J_i| > j_i^{(v)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(v)}}{k-1}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(v)}}{k-1}]}$$

Since $|J_i| \approx \left\|\left[\widetilde{b}, \frac{j_i + j_i^{(v)}}{k-1}\right]\right\|$, we conclude that

$$\|p_{k-1}(f)\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(k-1)}}{k-1}]} \le C(p,\mu)\|f\|_{L_{\psi,p}[\widetilde{b},\frac{j_i+j_i^{(k-1)}}{k-1}]} \quad \text{… (1.3.3}$$

From the above cases and since J_A super set in the interpolate set of J_i and by (1.3.2) and (1.3.3) then obtain

$$\|p_{k-1}(f)\|_{L_{\psi,p}(J_A)} \le C(p,\mu)\|f\|_{L_{\psi,p}(J_A)} .$$

Lemma 1.3.4. Let $f \in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$, then there exist $p_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ points inside ℓ_i, such that

$$\|f - p_{k-1}(f)\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{\psi,p} \quad \text{… (1.3.5}$$

Proof:

For an interval J_i , such that

$$J_i = \left[\frac{j_i + j_i^{(v)}}{k-1}, \frac{j_i + j_i^{(k-1)}}{k-1}\right], \quad \text{we have} \quad |J_i| = \frac{j_i^{(k-1)} - j_i^{(v)}}{k-1},$$

we denote $\ell_i \setminus J_i = \left(\left[j_i^{(v)}, \frac{j_i + j_i^{(v)}}{k-1} \right) \cup \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)} \right] \right)$.

Since $\ell_i = (k-1)J_i$, which means ℓ_i consists of $k-1$ interval J_i with $(k-1)|J_i| = |\ell_i|$, $k \ge 4$, therefore let

$$|J_i| \approx \left\|\left[j_i^{(v)}, \frac{j_i + j_i^{(v)}}{k-1} \right)\right\| \quad \text{also} \quad |J_i| \approx \left\|\left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)} \right]\right\| \quad \text{… (1.3.6}$$

It is sufficient to prove (1.3.5) for the interval ℓ_i , from the fact (1.2.1) assume $J_i \subset (k-1)J_i = \ell_i$, $k \ge 4$.

Now ,since $f \in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$,so by Lemma (1.3.1) there exist $p_{k-1}(f)$ interpolate f at k points inside J_i, hence we get from

$$\|p_{k-1}(f)\|_{L_{\psi,p}(J_i)} \le C(p,\mu)\|f\|_{L_{\psi,p}(J_i)}$$

since (1.3.6) are satisfy then we get , where $J_i' = \left[j_i^{(v)}, \frac{j_i + j_i^{(v)}}{k-1} \right)$ and $J_i'' = \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)} \right]$, that

$$\|p_{k-1}(f)\|_{L_{\psi,p}(J_i')} \le C(p,\mu)\|f\|_{L_{\psi,p}(J_i')}$$

$$\|p_{k-1}(f)\|_{L_{\psi,p}(J_i'')} \le C(p,\mu)\|f\|_{L_{\psi,p}(J_i'')} \ .$$

And from the fact that $\ell_i \setminus J_i = \left(\left[j_i^{(v)}, \frac{j_i + j_i^{(v)}}{k-1} \right) \cup \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)} \right] \right)$, we get

$$\|p_{k-1}(f)\|_{L_{\psi,p}(\ell_i \setminus J_i)} \le C(p,\mu)\|f\|_{L_{\psi,p}(\ell_i \setminus J_i)} \ .$$

Now , applied the same relation in (1.2.2) , for an interval J_i , we get

$$\|f - p_{k-1}(f)\|_{L_{\psi,p}(J_i)} \le C(p)\omega_k(f,|J_i|,J_i)_{\psi,p}$$

since $|J_i| \to 0$, then we get

$$\|f - p_{k-1}(f)\|_{L_{\psi,p}(J_i)} \le C(p)\omega_k^{\varphi}(f,|J_i|,J_i)_{\psi,p}$$

since $J_i \subset (k-1)J_i = \ell_i$, then we get

$$\|f - p_{k-1}(f)\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{\psi,p} .$$

Lemma 1.3.7. Let $f \in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$, $p < 1$, then there exist a polynomial $q_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ the points inside ℓ_i, such that

$$\|f - q_{k-1}(f)\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{\psi,p}$$

Proof:

By using Lemma (1.3.4), there exist a polynomial q_{k-1} of degree $\le k-1$ copositive with f in ℓ_i and interpolate f at the points inside ℓ_i, hence we have from (1.3.4)

$$\left\|f-q_{k-1}\right\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{\psi,p}$$

$$\left|f-q_{k-1}\right| \le C(p)\left\|f-q_{k-1}\right\|_{L_{\psi,p}(\ell_i)}$$

then we get

$$\left|f-q_{k-1}\right| \le C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{\psi,p}$$

Now by take $L_{\psi,p}(\ell_i)$-norm of both sides we obtain

$$\left\|f-q_{k-1}\right\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\left\|\omega_k^{\varphi}(f,|\ell_i|,\ell_i)\right\|_{L_{\psi,p}(\ell_i)}$$

By τ -modulus (or Sendove Popov modulus) with weight ψ, for f on ℓ_i, we get

$$\left\|f-q_{k-1}\right\|_{L_{\psi,p}(\ell_i)} \le C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{\psi,p}$$

1.4. PROOF OF THEOREM 1.1.8

Let $\mu>0$ be a fixed and let ℓ_i, $i=1,\dots,s$ be an interval of length $|\ell_i| = j_i^{(k-1)} - j_i^{(v)}$, $k>1$, $v=1,\dots,k-2$ in the center of $I=[-b,b]$, that is

$\mathrm{dis}(\ell_i,-b)=\mathrm{dis}(\ell_i,b)$, then by (1.3.4) there exist a linear polynomial $q_{k-1}^{*}\in\Pi_{k-1}$ copositive and interpolate f at k points in side $\ell_i \cap J_A$, hence we get

$$\left\|f-q_{k-1}^{*}\right\|_{L_{\psi,p}(I)} \le C(p,k)\omega_k^{\varphi}(f,|I|,I)_{\psi,p} \qquad \dots (1.4.1$$

Also by (1.3.4) there exist a linear polynomial $h_{k-1}=h_{k-1}(f)\in\Pi_{k-1}$ copositive and interpolate f at k points in side J_B , where $J_B=\left[b-\mu|I|, b-\frac{1}{2}\mu|I|\right]$, $\mu<\frac{1}{2}$ and $b-\frac{1}{2}\mu|I|\le b$, then

$$\left\|f-h_{k-1}(f)\right\|_{L_{\psi,p}(I)} = \left\|f-q_{k-1}^{*}+q_{k-1}^{*}-h_{k-1}(f)\right\|_{L_{\psi,p}(I)}$$

$$\le C(p)\left\|f-q_{k-1}^{*}\right\|_{L_{\psi,p}(I)} + C(p)\left\|q_{k-1}^{*}-h_{k-1}(f)\right\|_{L_{\psi,p}(I)}$$

$$= C(p)\left\|f-q_{k-1}^{*}\right\|_{L_{\psi,p}(I)} + C(p)\left\|h_{k-1}(f-q_{k-1}^{*})\right\|_{L_{\psi,p}(I)}$$

$$\le C(p)\left\|f-q_{k-1}^{*}\right\|_{L_{\psi,p}(I)} + C(p)\left\|h_{k-1}(f-q_{k-1}^{*})\right\|_{L_{\psi,p}(J_C)}$$

where $J_C = [b-\mu|I|, b]$, and $|J_B| \approx |J_C|$, since we have from an interval J_B and J_C that $b-\mu|I| \le b - \frac{1}{2}\mu|I|$, hence

$$\|f - h_{k-1}(f)\|_{L_{\psi,p}(I)} \le C(p)\|f - q_{k-1}^*\|_{L_{\psi,p}(I)} + C(p)\|h_{k-1}(f - q_{k-1}^*)\|_{L_{\psi,p}(J_B)}$$

by lemma (1.3.1) , we get

$$\|f - h_{k-1}(f)\|_{L_{\psi,p}(I)} \le C(p)\|f - q_{k-1}^*\|_{L_{\psi,p}(I)} + C(p)\|f - q_{k-1}^*\|_{L_{\psi,p}(J_B)}$$

by lemma (1.3.7) and inequality (1.4.1) we get

$$\|f - h_{k-1}(f)\|_{\psi,p} \le C(p,k)\omega_k^{\varphi}(f,|I|,I)_{\psi,p} \qquad \ldots (1.4.2$$

also there exist a linear polynomial $g_{k-1} \in \Pi_{k-1}$, copositive and interpolate f at k points in side J_A , where $-b+\mu|I| \ge -b$ also $b-\mu|I| \le b$, hence

$$\|f - g_{k-1}\|_{L_{\psi,p}(I)} = \|f - h_{k-1}(f) + h_{k-1}(f) - g_{k-1}(f)\|_{L_{\psi,p}(I)}$$

$$\le C(p)\|f - h_{k-1}\|_{L_{\psi,p}(I)} + C(p)\|g_{k-1} - h_{k-1}(f)\|_{L_{\psi,p}(I)}$$

$$= C(p)\|f - h_{k-1}\|_{L_{\psi,p}(I)} + C(p)\|g_{k-1}(f - h_{k-1})\|_{L_{\psi,p}(I)}$$

$$\le C(p)\|f - h_{k-1}\|_{L_{\psi,p}(I)} + C(p)\|g_{k-1}(f - h_{k-1})\|_{L_{\psi,p}(J_k)}$$

where $J_k = [-b+\mu|I|, b]$, and $|J_A| \approx |J_k|$, since we have from an interval J_A and J_k that $-b+\mu|I| \le b-\mu|I|$, hence

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \le C(p)\|f - h_{k-1}\|_{L_{\psi,p}(I)} + C(p)\|g_{k-1}(f - h_{k-1})\|_{L_{\psi,p}(J_A)}$$

by lemma (1.3.1) , we get

$$\|f - g_{k-1}\|_{L_{\psi,p}(I)} \le C(p)\|f - h_{k-1}\|_{L_{\psi,p}(I)} + C(p)\|f - h_{k-1}\|_{L_{\psi,p}(J_A)}$$

and by lemma (1.3.7) and inequality (1.4.2) we get

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \le c(p,k)\omega_k^{\varphi}(f,|I|,I)_{\psi,p} \qquad \ldots (1.4.3$$

(1.4.3) means there exist a polynomial copositive and interpolate f in an interval J_A , such that $-b+\mu|I| \le b$ and satisfy the Whitney theorem .

And by the same method in the above we can get the same result for an interval J_A such that $-b \le b-\mu|I|$.

Hence the result is true for I . If $\mu = 0$ then the inequality (1.4.3) is not true.

1.5. PROOF OF THEOREM 1.1.9

By using lemma (1.3.4) there exist a polynomial $g^* \in \Pi_{k-1} \cap \Delta^0(J_s)$ of degree $\le k-1$, and let g^* best approximation to f on $I=[-b,b]$, and let

$$\frac{f(t)-g^*(t)}{\psi(t)} < E_{k-1}(f,J_s)_{\psi,p} \quad ,$$

$$\frac{g^*(t)-f(t)}{\psi(t)} > -E_{k-1}(f,J_s)_{\psi,p}$$

$$\frac{g^*(t)}{\psi(t)} + E_{k-1}(f,J_s)_{\psi,p} > \frac{f(x)}{\psi(t)}$$

Let $f \in L_{\psi,p}(I) \cap \Delta^0(J_s)$, $p<1$, then for $k>1$ there exist a polynomial $p_{k-1} \in \Pi_{k-1}$ of degree $\le k-1$, such that

$$p_{k-1}(t) = \frac{g^*(t)}{\psi(t)} + E_{k-1}(f,J_s)_{\psi,p} > \frac{f(t)}{\psi(t)},$$

when $f(t) \ge 0$ and since $\psi(t+kh)$ nondecresing then $\frac{f(t)}{\psi(t)} \ge 0$, hence

$$\frac{g^*(t)}{\psi(t)} + E_{k-1}(f,J_s)_{\psi,p} > 0,$$

and when $f(t)<0$, then $\frac{f(t)}{\psi(t)} < 0$, hence

$$\frac{g^*(t)}{\psi(t)} + E_{k-1}(f,J_s)_{\psi,p} < 0$$, this implies that $p_{k-1}(t) \in \Delta^0(J_s)$ and

since $p_{k-1} \in \Pi_{k-1}$ then we get $p_{k-1}(t) \in \Pi_{k-1} \cap \Delta^0(J_s)$, this meaning p_{k-1} copositive with f at every points in an interval I .

Now ,

$$p_{k-1}(t) = \frac{g^*(t)}{\psi(t)} + E_{k-1}(f,J_s)_{\psi,p} > \frac{f(t)}{\psi(t)}$$

Since $p_{k-1}(t) \ge \frac{p_{k-1}(t)}{\psi(t)}$, then

$$p_{k-1}(t) - \frac{f(t)}{\psi(t)} \ge \frac{p_{k-1}(t)}{\psi(t)} - \frac{f(t)}{\psi(t)}$$

$$\frac{p_{k-1}(t)}{\psi(t)}-\frac{f(t)}{\psi(t)}\le\frac{g^*(t)}{\psi(t)}-\frac{f(t)}{\psi(t)}+E_{k-1}(f,J_s)$$

$$\frac{p_{k-1}(t)-f(t)}{\psi(t)}\le\frac{g^*(t)-f(t)}{\psi(t)}+E_{k-1}(f,J_s)$$

$$\int_I\left|\frac{p_{k-1}(t)-f(t)}{\psi(t)}\right|^p dt\le\int_I\left|\frac{g^*(t)-f(t)}{\psi(t)}\right|^p dt+E_{k-1}(f,J_s)_{\psi,p}$$

that is

$$\|f-p_{k-1}\|^p_{L_{\psi,p}(I)}\le\|f-g^*\|^p_{L_{\psi,p}(I)}+c(p)E_{k-1}(f,J_s)^p_{\psi,p}$$

Since $\|f-g^*\|_{L_{\psi,p}(I)}=E_{k-1}(f,J_s)_{\psi,p}$, then

$$\|f-p_{k-1}\|^p_{L_{\psi,p}(I)}\le C(p)\|f-g^*\|^p_{L_{\psi,p}(I)}\ .$$

By (1.3.4) there exist a polynomial $q_{k-1}(f)$ such that

$$g^*(t)\big|_{\ell_i}=q_{k-1}(f,t)\big|_{\ell_i}=q_{k-1}(f,t,j_1,...,j_s)\big|_{\ell_i},\quad k>1\ ,$$

where q_{k-1} , be a linear polynomial of degree $\le k-1$, interpolate f at the points $\{j_i\}_{i=1}^s$ inside ℓ_i, where $|\ell_i|=(k-1)|J_i|$, and $q_{k-1}\in\Pi_{k-1}\cap\Delta^0(\ell_i)$,then by using lemma (1.3.7) we have

$$\|f-q_{k-1}\|_{L_{\psi,p}(\ell_i)}\le C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{\psi,p}$$

$$\|f-g^*\|_{L_{\psi,p}(\ell_i)}\le C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{\psi,p}$$

then

$$\|f-g^*\|_{L_{\psi,p}(I)}\le C(p,k)\tau_k(f,|I|,I)_{\psi,p}$$

hence

$$\|f-p_{k-1}\|_{L_{\psi,p}(I)}\le C(p,k)\tau_k(f,|I|,I)_{\psi,p}\ .$$

المستخلص : في هذا البحث لدينا نتائج رئيسية حول أيجاد العلاقة بين درجة أفضل تقريب وبين المقياس τ – من الرتبة k في الفضاء $p<1$ $L_{\psi,p}(I),$ وان الدالة f تكون حافظة للإشارة عند النقاط في الفترة $I=[-b,b]$ وأيضا تم أيجاد العلاقة بين درجة أفضل تقريب بواسطة متعددة حدود درجتها اقل أو يساوي $k-1$ وبين مقياس النعومة ذو درجة اقل أو يساوي k للدالة $f \in L_{\psi,p}(I) \cap \Delta^0(J_s)$ ، $p<1$ المحافظة للإشارة .

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