BIFURCATION ANALYSIS AND CAUSTIC OF SIXTH ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract.

In this paper we studied the nonlinear elastic beams equation on elastic foundations. We used the local Lyapunov- Schmidt method to find The bifurcation equation and the Discriminant set. Also, we calculated the topological indices of the solutions of the problem.

Key Words: Lyapunov-Schmidt redaction, Wave Equation , Discriminant set.

1. Introduction.

The main goal in the following paper is the bifurcation solutions and Caustic of the nonlinear ODE given by:

$$\mathbb{z}^{(6)}(x) + \delta\mathbb{z}^{(4)}(x) + \alpha\mathbb{z}''(x) + \beta\mathbb{z} + \mathbb{z}\mathbb{z}'(x) = \Psi \qquad \dots (1.1)$$

with the following boundary conditions,

$$\mathbb{z}(0) = \mathbb{z}(\pi) = \mathbb{z}''(0) = \mathbb{z}''(\pi) = \mathbb{z}^{iv}(0) = \mathbb{z}^{iv}(\pi) = 0,$$

where $\mathbb{z}$ is the deflection of beam and $\Psi = \varepsilon\varphi(x)$ (ε - small parameter) is a continuous function. Equation (1.1) is a special case of the general nonlinear differential equation of sixth order,

$$\mathbb{z}^{(6)}(x) + \delta\mathbb{z}^{(4)}(x) + \alpha\mathbb{z}''(x) + \beta\mathbb{z} + \mathbb{N}(\lambda, \tilde{\mathbb{z}}) = \Psi, \qquad \dots (1.2)$$

$$\tilde{\mathbb{z}} = \left(\mathbb{z}, \mathbb{z}', \mathbb{z}'', \mathbb{z}''', \mathbb{z}^{iv}, \mathbb{z}^{v}, \mathbb{z}^{vi}\right).$$

An analogous problem of fourth order of equation (1.2) has been studied by Sapronov [16] applied the local method of Lyapunov-Schmidt

(LMLS) and found the bifurcation solutions of equation (1.2) when $\mathbb{N}(\lambda,\tilde{\mathbb{z}}) = \mathbb{z}^3$ and $\Psi = 0$ with the boundary conditions,

$$\mathbb{z}(0) = \mathbb{z}(\pi) = \mathbb{z}''(0) = \mathbb{z}''(\pi) = 0,$$

in his study he solved the bifurcation equation corresponding to the equation (1.2) and found the bifurcation diagram of a specify problem. When $\mathbb{N}(\lambda,\mathbb{z}) = \mathbb{z}^2$ and $\Psi \neq 0$, equation (1.2) has been studied by Abdul Hussain [7], it was shown that by using LMLS the existence of bifurcation solutions of equation (1.2) with the conditions,

$$\mathbb{z}(0) = \mathbb{z}(1) = \mathbb{z}''(0) = \mathbb{z}''(1) = 0,$$

and in another study with the conditions,

$$\mathbb{z}(x_1) \geq \varepsilon_1, \quad \mathbb{z}(x_2) \geq \varepsilon_2,$$

$$0 < x_1 < x_2 < 1, \ \varepsilon_1, \varepsilon_2 \quad \text{are small parametrs,}$$

also, a new study of corner singularities of smooth maps was given in the analysis of bifurcations balance of the elastic beams and periodic waves.

When $\mathbb{N}(\lambda,\tilde{\mathbb{z}}) = \mathbb{z}^2 + \mathbb{z}^3$ equation (1.2) was studied by Sapronov [19], he found bifurcation periodic solutions of equation (1.2) by LMLS. Also, he solved the bifurcation equation corresponding to the equation (1.2) and found the bifurcation diagram of a specify problem. When $\mathbb{N}(\lambda,\mathbb{z}) = -\wp\mathbb{z}^3$ and $\Psi = 0$ equation (1.2) has been studied as follows: Furta and Piccione [4] showed the existence of periodic travelling wave solutions of equation (1.2) describing oscillations of an infinite beam, which lies on a non-linearly elastic support with non-small amplitudes. Thompson and Stewart [6] showed numerically the existence of periodic solutions of equation (1.2) for some values of parameters. Bardin and Furta [1] used the LMLS and found the sufficient conditions of existence of periodic waves of equation (1.2). Also, they introduced the solutions of equation (1.2) in the form of power series.

When $\mathbb{N}(\lambda,\tilde{\mathbb{Z}}) = \mathbb{z}^2 + \mathbb{z}^3$ and $\Psi \neq 0$ equation (1.2) was studied by Mohammed [10], he found bifurcation solutions of equation (1.2) by using

LMLS. Also, he solved the bifurcation equation corresponding to the equation (1.2) and found the bifurcation diagram of a specify problem. When $\mathbb{N}(\lambda,\tilde{\mathbb{z}}) = \mathbb{z}^2 + \mathbb{z}^3$, $\delta = 1$, $\Psi \neq 0$ and $\varepsilon_1, \varepsilon_2 \neq 0$ equation (1.2) was studied by Abdul Hussain [8], he found the bifurcation solutions of equation (1.2) by using LMLS. When $\mathbb{N}(\lambda,\tilde{\mathbb{z}}) = \mathbb{z}\mathbb{z}^{//}$ and $\Psi \neq 0$ equation (1.2) was studied by Mohammed [11], he found bifurcation solutions of equation (1.2) applied the LMLS. Also, he found the Discriminant set (bifurcation set) of the elastic beams equation for some values of parameters. When $\mathbb{N}(\lambda,\mathbb{z}) = \mathbb{z}^2$, $\Psi = 0$ and $\delta = 1+\rho$, ρ is small parameter, equation (1.2) was studied by Shanan and Abdul Hussain[14], they showed that the bifurcation equation corresponding to the elastic beams equation is given by nonlinear system of three quadratic equations. Also, they determined the bifurcation diagram of the specified problem. When $\mathbb{N}(\lambda,\mathbb{z}) = \mathbb{z}^2$, $\Psi \neq 0$ and $\delta = 1+\rho$, ρ is small parameter, equation (1.2) was studied by Abdul Hussain[9], he showed that the bifurcation equation corresponding to the elastic beams equation is given by nonlinear system of three quadratic equations. Also, he completely determined the bifurcation diagram of the specified problem. When $\mathbb{N}(\lambda,\mathbb{z}) = \mathbb{z}^2\mathbb{z}'$, $\Psi \neq 0$ equation (1.2) was studied by Mohammed [13], he found bifurcation solutions of equation (1.2). Also, he found bifurcation set of the elastic beams equation for some values of parameters. Also, he calculated the topological indices of the solutions of the following nonlinear system of algebraic equations

2. Preliminaries

In this section, we review some definitions and basic properties to show our main results.

Definition 2.1[19]

Let X and Y be real linear Banach spaces . Let $A: X \to Y$ be a linear continuous operator . Then A is a Fredholm operator if the space $Ker(A)$ and $Coker(A) = Y/Im(A)$ are finite dimensional . The number $ind(A) = dim\, Ker(A) - dim\, Coker(A)$ is called the (Fredholm) index of the operator A.

The many equations that appear in the Mathematical and physical problems are the equations of Fredholm with parameters,

$$L(x,\lambda) = b, \quad x \in O \subset X, \; b \in Y, \; \lambda \in R^n. \qquad \text{.... (2.1)}$$

where L is a smooth Fredholm mapping of zero index between Banach spaces X and Y and O is an open subset in Y. The solutions of these problems can be found by solving the equivalent equation, [19]

$$\Phi(\xi,\lambda) = \beta, \quad \xi \in \hat{M}, \quad \beta \in \hat{N}, \qquad \text{.....(2.2)}$$

where $\hat{M}$ and $\hat{N}$ are smooth n-dimensional manifolds. Vainberg, and Trenogin [12], Loginov[2] and Sapronov[17,18] are dealing with equation (2.1) into the equation (2.2).

Definition 2.2[3]

Let $\mathcal{H}: R^m \to R^n$ be a a smooth map. A point $x_0 \in R^m$is called a regular point of $\mathcal{H}$ if the jacobian $D\mathcal{H}$ has maximal rank $\min\{m, n\}$. A value $y_0 \in R^n$ is called a regular value of $\mathcal{H}$ if x_0 is a regular point of $\mathcal{H}$ for all $x \in \mathcal{H}^{-1}(y_0)$. Points and values are called singular if they are not regular.

Definition 2.3[19]

The set of all λ in which equation (2.1) has degenerate solutions is called the Discriminant set(bifurcation set) of equation (2.1), denoted by $\sum$.

Definition 2.4 [19]

The Discriminate set together with the spreading of the regular solutions in the space of parameters is called the Bifurcation diagram (Caustic).

Theorem 1.1 [19]

The point $s \in X$ is a solution of the equation $L(x,\lambda) = 0$ if and only if, $s = \sum_{i=1}^{n} \overline{x}_i e_i + \Theta(\overline{\eta}, \overline{\lambda})$, where, $\overline{\eta}$ is a solution of equation $\phi(\eta, \widetilde{\lambda}) = 0$, $\eta = (x_1, x_2, ..., x_n)$.

3. Problem Analysis Using LMLS

The LMLS [19, 5] relies on Fredholm operators of finite index defined in Banach spaces which are immerged in appropriate Hilbert spaces in order to use two projections allowing to transform an infinite dimensional problem into a finite one. Following this, a bifurcation equation is established, solving this equation gives us the number of solutions of the problem.

In this paper we studied the bifurcation solutions and Caustic of boundary value problem when $\delta = 1$,

$$\mathbb{z}^{(6)}(x) + \mathbb{z}^{(4)}(x) + \alpha\mathbb{z}''(x) + \beta\mathbb{z} + \mathbb{z}\mathbb{z}'(x) = \Psi \qquad \ldots.(3.1)$$

$$\mathbb{z}(0) = \mathbb{z}(\pi) = \mathbb{z}''(0) = \mathbb{z}''(\pi) = \mathbb{z}^{iv}(0) = \mathbb{z}^{iv}(\pi) = 0,$$

3.1. **Reduction to the Bifurcation Equation**

The LMLS is efficient method for study problem (3.1). By reduction equation in (3.1) into finite dimensional system of two nonlinear equations, the method allowed us to study the bifurcation solutions of this equation.

Now we prove the reduction theorem of problem (3.1).

Theorem 3.1

The bifurcation equation corresponding to the problem (3.1) is given by the following nonlinear system of two equations,

$$\Phi(\xi,\widetilde{\lambda}) = \begin{pmatrix} r\xi_1\xi_2 + \lambda_1\xi_1 - q_1 \\ r\xi_1^2 + \lambda_2\xi_2 \end{pmatrix} + o(|\xi|^2) + O(|\xi|^2)O(\delta) = 0.$$

where, $\xi = (\xi_1, \xi_2)$, $\widetilde{\lambda} = (\lambda_1, \lambda_2, q_1) \in R^3$, $\delta = (\delta_1, \delta_2)$.

Furthermore, the solutions of the equation in (3.1) are in one-to-one correspondence with the solutions of the nonlinear system.

Proof:

Step 1. Let

$$L(\mathbb{z},\lambda) = \mathbb{z}^{(6)}(x) + \mathbb{z}^{(4)}(x) + \alpha\mathbb{z}''(x) + \beta\mathbb{z} + \mathbb{z}\mathbb{z}'(x) \qquad \ldots..(3.2)$$

where $L: X \rightarrow Y$ is a nonlinear Fredholm map of index zero from Banach space X to Banach space Y, $X = \{\mathbb{z}(x) \in C^6([0,\pi], R): \mathbb{z}(0) = \mathbb{z}(\pi) = \mathbb{z}''(0) = \mathbb{z}''(\pi) = \mathbb{z}^{iv}(0) = \mathbb{z}^{iv}(\pi) = 0,\}$, $Y = \{\mathbb{z}(x) \in C^0([0,\pi], R)\}$ and

$\mathbb{z} = \mathbb{z}(x),\ \lambda = (\alpha, \beta)$. In this case, the bifurcation solutions of equation (3.1) is equivalent to the bifurcation solutions of the operator equation,

$$L(\mathbb{z}, \lambda) = \Psi, \quad \Psi \in Y. \qquad \dots (3.3)$$

Since, the Fre'chet derivative of the nonlinear operator $L(\mathbb{z}, \lambda)$ at the point $(0, \lambda)$ is

$$dL(0, \lambda) = \mathcal{L}^{vi} + \mathcal{L}^{iv} + \alpha \mathcal{L}^{xx} + \beta \mathcal{L}, \qquad \mathcal{L} \in X$$

Step 2. The linearized equation corresponding to the equation (3.1) is given by,

$$A\mathcal{L} = 0, \qquad \mathcal{L} \in X,$$

$$A = dL(0, \lambda) = \frac{d^6}{dx^6} + \frac{d^4}{dx^4} + \alpha \frac{d^2}{dx^2} + \beta, \qquad x \in [0, \pi]$$

$$\mathcal{L}(0) = \mathcal{L}(\pi) = \mathcal{L}''(0) = \mathcal{L}''(\pi) = \mathcal{L}^{iv}(0) = \mathcal{L}^{iv}(\pi) = 0.$$

The solutions of linearized equation which satisfied the boundary conditions is given by,

$$e_p(x) = c_p \sin(px), \qquad p = 1, 2, 3, \dots \qquad \dots .(3.4)$$

Substitute (3.4) in the linearized equation we have characteristic equation corresponding to this solution in the form,

$$-p^6 + p^4 - \alpha p^2 + \beta = 0 .$$

This equation gives in the $\alpha\beta$-plane characteristic lines ℓ_p. The characteristic lines ℓ_p consists the points (α, β) for which the linearized equation has non-zero solutions. The point of intersection of characteristic lines in the $\alpha\ \beta$-plane is a bifurcation point [19]. So for equation (3.2) the point $(\alpha, \beta) = (-10, -10)$ is a bifurcation point. Localized parameters α, β as follows,

$$\alpha = -10 + \delta_1, \qquad \beta = -10 + \delta_2, \qquad \delta_1, \delta_2 \text{ are small parameters,}$$

which leads to bifurcation along the modes

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad k = 1,2.$$

Step 3. Decompose

$$\begin{aligned} &(a) \;\; X = N \oplus X^{\infty-2} \quad \textit{and} \\ &(b) \;\; Y = \widetilde{N} \oplus Y^{\infty-2}, \end{aligned} \qquad \text{.....(3.5)}$$

where $N = Ker(A) = span\ \{e_1, e_2\}$ and $\widetilde{N} = Range(A)$ respectively.

Step 4. Use (3.5a) every $\mathbb{z} \in X,\ \exists n \in N,\ \exists y \in X^{\infty-2} : \mathbb{z} = n + y,$

where
$$n = \sum_{i=1}^{2} \xi_i\, e_i \in N, \quad N \perp y \in X^{\infty-2}.$$

Step5. Let us define two projections $\Pi_1 : Y \longrightarrow \widetilde{N}$ and $\Pi_2 : Y \longrightarrow \widetilde{N}^{\perp} = Y^{\infty-2}$
such that

$$\Pi_1 L(\mathbb{z}, \lambda) = L_1(\mathbb{z}, \lambda)$$
$$\Pi_2 L(\mathbb{z}, \lambda) = L_2(\mathbb{z}, \lambda)$$

and hence,

$$L(\mathbb{z}, \lambda) = L_1(\mathbb{z}, \lambda) + L_2(\mathbb{z}, \lambda)$$

$$L_1(\mathbb{z}, \lambda) = \sum_{i=1}^{2} y_i(\mathbb{z}, \lambda) e_i \in N, \quad L_2(\mathbb{z}, \lambda) \in \widetilde{N}^{\perp},$$
$$y_i(\mathbb{z}, \lambda) = \langle L(\mathbb{z}, \lambda), e_i \rangle.$$

Equation (3.3) is equivalent to the following system,

$$\begin{cases} L_1(\mathbb{z}, \lambda) = \Psi_1, \\ L_2(\mathbb{z}, \lambda) = \Psi_2 \end{cases}$$

or

$$\begin{cases} L_1(n+y,\lambda) = \Psi_1, \\ L_2(n+y,\lambda) = \Psi_2 \end{cases}$$

where $\Psi = \Psi_1 + \Psi_2$, $\quad \Psi_1 = \sum_{i=1}^{2} q_i\, e_i \in \widetilde{N}$ and $\Psi_2 \in Y^{(\infty-2)}$.

Step 6. By implicit function theorem, there exists a smooth map $\Theta: N \longrightarrow N^{\perp}$ (depending on λ), such that $\Theta(n,\lambda) = y$ and

$$L_2(n + \Theta(n,\lambda),\lambda) = \Psi_2$$

To find the solutions of the equation (3.3) in the neighbourhood of the point $\mathbb{z} = 0$ it is sufficient to find the solutions of the equation,

$$L_1(n + \Theta(n,\lambda),\lambda) = \Psi_1 \quad \text{(bifurcation equation)}$$

We have the bifurcation equation in the form,

$$\Phi(\xi,\lambda) = \Psi_1, \quad \xi = (\xi_1,\xi_2), \quad \lambda = (\alpha,\beta),$$

where,

$$\Phi(\xi,\lambda) = L_1(n + \Theta(n,\lambda),\lambda) \quad .$$

Step 7. Equation (3.3) can be written in the form,

$$\begin{aligned} L(n+y,\lambda) &= A(n+y) + \mathbb{N}(n+y) \\ &= An + nn' + \cdots \end{aligned}$$

where $\mathbb{N}(n+y) = (n+y)(n' + y')$ and the dots denotes the terms that consists the element y together. Hence,

$$\begin{aligned} \Phi(\xi,\lambda) &= L_1(n + \mathrm{y},\lambda) \\ &= \sum_{i=1}^{2} \langle An + nn', e_i \rangle e_i + \cdots = \Psi_1 \qquad \ldots(3.6) \end{aligned}$$

where ($\langle \cdot,\cdot \rangle$ is the inner product in Hilbert space $\mathbf{L}_2([0,\pi], R)$).

Equation (3.6) implies that,

$$\langle An + nn', e_1\rangle e_1 + \langle An + nn', e_2\rangle e_2 + \cdots = q_1 e_1 + q_2 e_2 \qquad \ldots (3.7)$$

Some calculations of equation (3.7) implies that,

and

$$\langle An + nn', e_2\rangle = \lambda_2 \xi_2 + \mathrm{r}\,\xi_1^2$$

where,

$$\lambda_1 = Ae_1 = \alpha_1(\lambda)e_1,\ \lambda_2 = Ae_2 = \alpha_2(\lambda)e_2,\ \mathrm{r} = \frac{\sqrt{2}}{\sqrt{\pi}}$$

Step 8. The symmetry of the function $\Psi(x)$ with respect to the involution $I : \Psi(x) \mapsto \Psi(\pi - x)$ implies that q_2=0 and hence we have,

$$(r\xi_1\xi_2 + \lambda_1\xi_1)e_1 + (r\xi_1^2 + \lambda_2\xi_2)e_2 + \ldots. = q_1 e_1.$$

By equating the coefficients of e_1 and e_2 we have,

$$\Phi(\xi,\widetilde{\lambda}) = \begin{pmatrix} r\xi_1\xi_2 + \lambda_1\xi_1 - q_1 \\ r\xi_1^2 + \lambda_2\xi_2 \end{pmatrix} + o(|\xi|^2) + O(|\xi|^2)O(\delta) = 0 \ \square \qquad \ldots.(3.8)$$

3.2 Bifurcation Analysis and Caustic with Symmetry of $\Psi(x)$.

From theorem (1.1) the point $s \in X$ is a solution of equation (3.3) if and only if :

$$s = \sum_{i=1}^{2} \overline{\xi}_i e_i + \Phi(\overline{\xi}, \overline{\lambda})$$

where, $\overline{\xi}$ is a solution of equation

$$\phi(\xi, \widetilde{\lambda}) = 0, \qquad \ldots(3.9)$$

where

$$\phi(\xi,\tilde{\lambda})=\begin{pmatrix} r\xi_1\xi_2+\lambda_1\xi_1-q_1 \\ r\xi_1^2+\lambda_2\xi_2 \end{pmatrix}. \qquad \tilde{\lambda}=(\lambda_1,\lambda_2,q_1)$$

The system(3.9) has all the topological and analytical properties of equation (3.1). So to study bifurcation solutions of equation (1.1) it is sufficient to the study bifurcation solutions of system (3.9). The Discriminant set Σ of the map $\Phi(\xi,\tilde{\lambda})$ is locally equivalent in the neighbourhood of point zero to the Discriminant set of the map $\phi(\xi,\tilde{\lambda})$ [15]. This means that, to study the Discriminant set of the map $\Phi(\xi,\tilde{\lambda})$ it is sufficient to study the Discriminant set of the map $\phi(\xi,\tilde{\lambda})$.

In many applications the Discriminant set (Caustic) can be solved by finding a relationship between the parameters and variables given in the problem, but in some problems there is a difficulty for finding this parameterization. The second way for finding the Discriminant set it is by finding the parameter equation, that is; equation of the form

$$h(\tilde{\lambda})=0, \quad \tilde{\lambda}=(\lambda_1,\lambda_2,q_1)\in R^3.$$

such that the set of all $\tilde{\lambda}=(\lambda_1,\lambda_2,q_1)$ in which equation (3.9) has degenerate solutions that satisfy the equation $h(\tilde{\lambda})=0$,where $h:R^3\to R$ is a map. For equation (3.9) it is not easy to find a relationship between the parameters and variables given in the equation, so the second way has been used to determine the Discriminant set of equation (3.9). Thus, we have the following result,

Theorem 3.2 Caustic (the Discriminant set) of equation (3.9) in the space of parameters $(\lambda_1,\lambda_2,q_1)$ is the following surface

$$\pi\,\lambda_2\,(8\pi\,\lambda_1^3\,\lambda_2+27\,q_1^2).$$

The last theorem can be proved by solving the following system in terms of q_1, λ_1, λ_2. The system is,

$$\left.\begin{aligned}
&\lambda_1\xi_1-\frac{\sqrt{2}\,\xi_1\xi_2}{2\sqrt{\pi}}-q_1=0,\\
&\frac{\sqrt{2}\,\xi_1^2}{2\sqrt{\pi}}+\lambda_2\,\xi_2=0,\\
&\frac{\xi_1^2}{\pi}-\frac{\lambda_2\,\xi_2}{\sqrt{2\pi}}+\frac{2}{5}\lambda_1\,\lambda_2=0.
\end{aligned}\right)$$

To study the Discriminant set of equation (3.9) it is convenient to fix the value of one parameters. The sections of Discriminant set in some planes were described in the figures (1), (2), (3), (4) and (5). (All figures were found by using Maple 13).

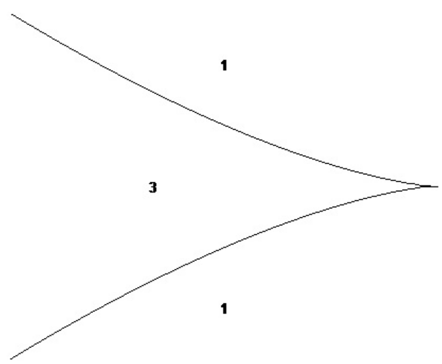

Fig. 1: Described Caustic in the $\lambda_1 q_1$ - plane when $\lambda_2 > 0$

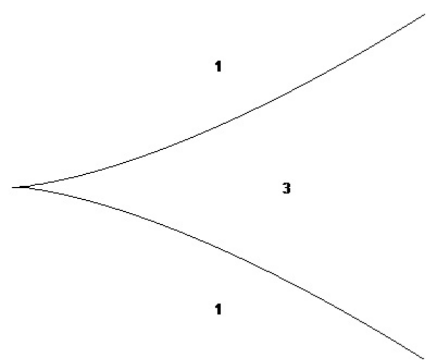

Fig. 2: Described Caustic in the $\lambda_1 q_1$ - plane when $\lambda_2 < 0$

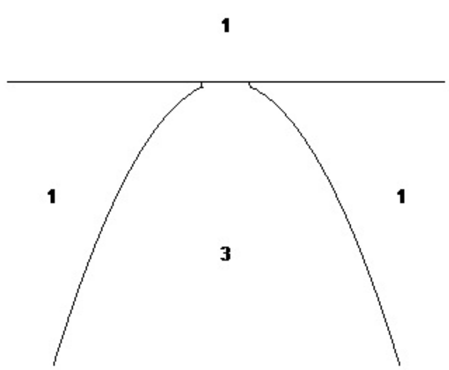

Fig. 3: Described Caustic in the $q_1 \lambda_2$ - plane when $\lambda_1 > 0$

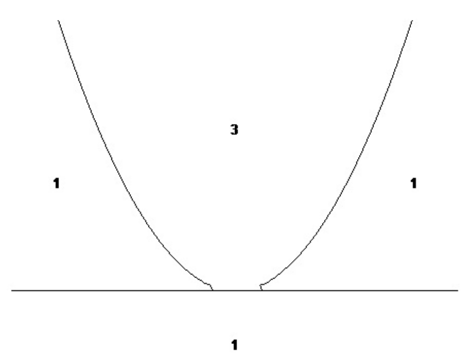

Fig. 4: Described Caustic in the $q_1 \lambda_2$ - plane when $\lambda_1 < 0$

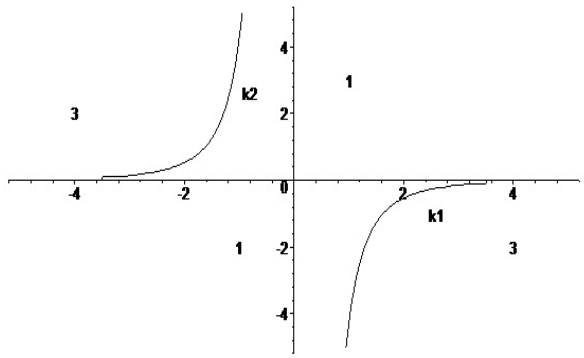

Fig. 5: Described Caustic in the $\lambda_1\lambda_2$ - plane when $q_1 > 0 \quad or \quad q_1 < 0$

Figure (1) and (2) describes the Caustic in the $\lambda_1 q_1$-plane when λ_2 >0 and λ_2<0. The number of regular solutions in every region was found is either one or three. In figure (3) and (4) describes the Caustic in the $q_1\lambda_2$-plane when λ_1>0 and λ_1<0 respectively. The number of regular solutions in every region was found is either one or three. Figure (5) describes the Caustic in the $\lambda_1\lambda_2$-plane when q_1 >0 or q_1<0. The number of regular solutions in every region was found is either one or three.

In figures (1), (2), (3), (4), and (5) the complement of the Discriminant set $\Omega = R^3 \setminus \Sigma$ is the union of two open subsets $\Omega = \Omega_1 \cup \Omega_2$ such that if $\tilde{\lambda} \in \Omega_1$ then equation (3.9) has one regular solutions with topological indices -1, if $\tilde{\lambda} \in \Omega_2$ then equation (3.9) has three regular solutions with topological indices -1,1,-1. Bifurcation Analysis and Caustic without Symmetry of $\Psi(x)$ in other paper.

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الخلاصة

في هذا البحث درسنا معادلة الأنابيب المرنة الغير خطية بالاعتماد على أساس المرونة ، استخدمنا طريقة (Lyapunov- Schmidt) المحلية لإيجاد معادلة التفرع المقابلة لمعادلة الأنابيب المرنة والمجموعة المميزة. كذلك تم حساب(topological indices) لحلول المسألة.