# On Maps Of Period 2 On Prime Near – Rings

**Abdul Rahman H. Majeed, Enaam F. Adhab**

Department of Mathematics , college of science , University of Baghdad . Baghdad , Iraq

**Abstract**

In this paper , we introduce the notion of maps of period 2 on near-ring N, the main our purpose is to study and investigate the existence and properties of mapping such as homomorphisms, anti - homomorphisms, α -derivations and α-centeralizers when they are of period 2 on near – rings.

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**Keywords:** prime near-ring, semiprime near-ring, mapping of period 2, homomorphisms, anti –homomorphisms, α -derivations and α-centeralizers.

## 1.Introduction

A right near – ring (resp.left near ring) is a set N together with two binary operations (+) and (.) such that (i)(N,+) is a group (not necessarily abelian).(ii)(N,.) is a semi group.(iii)For all a,b,c ϵ N ; we have (a+b).c = a.c + b.c (resp. a.(b+c) = a.b + b.c . Through this paper, N will be a zero symmetric left near – ring (i.e., a left near-ring N satisfying the property 0.x=0 for all x∈ N). we will denote the product of any two elements x and y in N ,i.e.; x.y by xy . The symbol Z will denote the multiplicative centre of N, that is Z=$\{x \in N \mid xy = yx \ for \ all \ y \in N\}$. N is called a prime near-ring if xNy = {0} implies either x = 0 or y = 0. It is called semiprime if xNx={0} implies x=0 . Near-ring N is called n-torsion free if nx=0 implies x=0. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal ) if NU⊆U (resp.UN⊆U)and if U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. A normal subgroup (I,+) of (N, +) is called a right ideal (resp. left ideal) of N if (x + i)y – xy ∈ I for all x,y ∈ N and i ∈ I(resp. xi ∈ I for all i∈I and x∈N ). I is called ideal of N if it is both a left ideal as well as a right ideal of N . For terminologies concerning near-rings , we refer to Pilz [7].

The concept of mapping of period 2 has already introduced in ring by Bell . H.E and Daif . M.N in [5], in the present paper, motivated by this concept we define a mapping of period 2 on near-ring . A mapping of the near ring N into itself is of period 2 on N if $f^2(x) = x$ for all x∈ N. Let U be a non empty subset of N, a mapping f:N → N is of period 2 on U if $f^2(x) = x$ for all x∈ U.

An additive mapping f:N → N is said to be homomorphism(anti-homomorphism) if f(xy)=f(x)f(y) for all x,y ∈ N (f(xy) = f(y)f(x) for all x,y ∈ N).Let U be a non empty subset of N, an additive mapping f:N → N is said to be homomorphism(anti-homomorphism) on U, if $f(xy)=f(x)f(y)$ for all $x,y \in U$ ($f(xy) = f(y)f(x)$ for all $x,y \in U$). Anti-homomorphism of period 2 in ring theory is called an involution and the ring admitting an involution * is called ring with involution * or * ring (see [2],[3],[5] ,[6] for reference where further references can be found). Note that when a homomorphism (anti-homomorphism) of period 2 then f is bijective .

Let $\alpha$ be an automorphism of N. An additive endomorphism $d : N \longrightarrow N$ is said to be an $\alpha$ -derivation of N if $d(xy) = \alpha(x)d(y) + d(x)y$ , or equivalently , as noted in( [8] , proposition 1) that $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x,y \in N$ .

The concept of $\alpha$ –centralizer has been studied in rings by Ali .S in [1] and Ashraf .M in [3] . In the present paper , motivated by this concept ,we define $\alpha$ –centralizer in near-rings and study some properties involved there. Let $\alpha$ be an automorphism of N . An additive endomorphism $d : N \longrightarrow N$ is said to be left $\alpha$ –centralizer (right $\alpha$ –centralizer)of N if $d(xy) = d(x)\alpha(y)$,($d(xy)=\alpha(x)d(y)$) for all $x,y \in N$ .

Many authors studied the relationship between structure of near – ring N and the behaviour of special mapping on N(see [2],[4],[6],[8] for reference where further references can be found) . In the year 2014 Bell . H.E and Daif . M.N in [5] studied the existence of derivations of period 2 on prime and semiprime rings . This research has been motivated by their works , we have extended their results in the setting of mappings of period 2 on a prime and semiprime near-ring .

## 2. Preliminaries.

The following lemmas are essential for developing the proofs of our main results.

**Lemma 2.1 [8]** Let d be an $\alpha$ -derivation on a near-ring N. Then

(i) $(\alpha(x)d(y) + d(x)y)z = \alpha(x)d(y)z + d(x)yz$ ;

(ii) $(d(x)y + \alpha(x)d(y))z = d(x)yz + \alpha(x)d(y)z$ . for all $x , y , z \in N$.

**Lemma 2.2 [8]** Let d be an $\alpha$ -derivation of a prime near-ring N and $a \in N$ such that $ad(x) = 0$ (or $d(x)a = 0$) for all $x \in N$ **.**Then a=0 or d=0 .

**Lemma 2.3 [8]** If U is a non-zero semigroup right ideal (resp, semigroup left ideal) and x is an element of N such that $Ux = \{0\}$ (resp, $xU = \{0\}$ ) then x = 0.

**Lemma 2.4** Let N be left near – ring , d is $\alpha$-derivation of N then

$$2\,(d(x)y + \alpha(x)d(y))z = 2d(x)yz + 2\,\alpha(x)d(y)z$$

**Proof .** By Lemma 2.1 (ii) we have

$$\begin{aligned} 2(d(x)y + \alpha(x)d(y))z &= 2(d(x)yz + \alpha(x)d(y)z) \\ &= d(x)yz + \alpha(x)d(y)z + d(x)yz + \alpha(x)d(y)z \\ &= d(x)yz + (\alpha(x)d(y) + d(x)y)z + \alpha(x)d(y)z \\ &= d(x)yz + (d(x)y + \alpha(x)d(y))z + \alpha(x)d(y)z \end{aligned}$$

$= d(x)yz + d(x)yz + \alpha(x)d(y)z + \alpha(x)d(y)z$

$= 2d(x)yz + 2\,\alpha(x)d(y)z$ .

## 3. Main Result.

In this section we study homomorphisms, anti-homomorphisms, α-derivations

and α–centralizers when they are of period 2.

**Theorem 3.1.** Let N be prime near – ring, U is a non-zero semigroup ideal of N, if U admits an endomorphism f, such that f is of period 2 and f(xy) = f(yx) for all x , y ϵ U, f(U) ⊆ U, then N is commutative ring.

**Proof.** $xy = f^2(xy) = f(f(xy)) = f(f(yx)) = f(f(y)f(x)) = f^2(y)f^2(x) = yx$ for all x , y ϵ U.

For all x , y ϵ U and r,s ϵ N we have xy (rs-sr) = xyrs – xysr = yxrs – ysxr = xrys – xrys = 0, this implies Uy(rs-sr) = {0}, by Lemma 2.3 we get y(rs-sr) = 0. i.e.;

U(rs-sr) = 0, using Lemma 2.3 again we conclude that rs = sr for all r,s ϵN. So we get

$(x+y)z = z(x+y) = zx + zy = xz + yz$ for all x , y ,zϵ N (1)

Now let x , y , zϵ N, by applying (1) we obtain

$(z+z)(x+y) = z(x+y)+z(x+y)$

$= zx + zy + zx + zy$ for all x , y ,z ϵ N.

On the other hand

$(z+z)(x+y) = (z+z)x + (z+z)y$

$= zx + zx + zy + zy$ for all x , y ,z ϵ N.

By comparing both expressions,we obtain

$zx + zy = zy + zx$ for all x , y ,z ϵ N.

This is reduced to $z(x + y – x - y) = 0$ for all x , y ,z ϵ N , it is mean that $N(x + y – x - y) = 0$, Lemma 2.3 implies that $x + y – x – y = 0$ for all x , y ϵ N, thus we conclude that N is a commutative ring.

**Corollary 3.1.** Let N be prime near – ring ,if N admits an endomorphism f, such that f is of period 2 and f(x y) = f(y x) for all x , y ϵ N , then N is commutative ring **.**

In the year 2013 it was proved by Boua and Raji [6 ,theorem 1] that if a prime near ring N admits an involution then N is a ring, we have extended this result in the setting of an anti-homomorphism of period 2( involution ) on a semiprime near-ring.

**Theorem(2.3).** Let N be a near – ring , U is a non-zero semigroup ideal of N, if N admits an anti-homomrphism f, such that f is of period 2 on U then the multiplicative law of N is right distributive .

**Proof.** $(x + y)z = f^2(x + y)f^2(z) = f(f(x + y))f(f(z)) = f(f(z)f(x + y)) = f(f(z)(f(x) + f(y))) = f(f(z)f(x) + f(z)f(y)) = f(f(z)f(x)) + f(f(z)f(y)) = (f(f(xz)) + f(f(yz)) = f^2(xz) + f^2(yz) = xz + yz$.

Thus we conclude that $(x + y)z = xz + yz$ for all $x,y,z \in U$ (2)

Replacing x and y by xt and xr respectively in (2) we obtain

$(xt + xr)z = xtz + xrz$ for all $x,z \in U$ and $t ,r \in N$, which can be written as $x(t + r)z = x(tz + rz )$ for all $x , z \in U , t , r \in N$ , that is mean $x((t + r)z - (tz + rz )) = 0$ for all $x ,z \in U , t, r \in N$, hence $U((t + r)z - (tz + rz )) = 0$ ,then by Lemma 2.3 we have

$(t + r)z = (tz + rz )$ for all $z \in U, t,r \in N$. (3)

Now replacing z by kz, where $k \in N$, in (3) we obtain

$(t + r)kz = tkz + rkz$ for all $z \in U , t ,r ,k \in N$ . (4)

By (3)we have $(tkz + rkz) = (tk + rk)z$ for all $z \in U , t ,r , k \in N$, thus relation (4) becomes

$(t + r)kz =(tk + rk)z$ for all $z \in U , t , r, k \in N$ . (5)

On the other hand, by (2) we have for all $z \in U , t ,r , k \in N$

$((t + r)k - (tk + rk))z = (t + r)kz + (- (tk + rk))z$ . (6)

Then by combining (5) and (6) ,we find that $((t + r)k - (tk + rk))z = (tk + rk)z + ( -(tk + rk))z = ((tk + rk) +( - (tk + rk))) z = 0.z = 0$, that is mean $((t + r)k - (tk + rk))U = 0$ for all $t, r , k \in N$, by Lemma 2.3 we get

$(t + r)k = tk + rk$ for all $t ,r ,k \in N$ . (7)

Hence, the multiplicative law of N is right distributive.

**Theorem 3.3.** Let N be semiprime near – ring, U is a non-zero semigroup ideal of N, if N admits an anti-homomrphism f, such that f is of period 2 on U then N is a ring.
**Proof .** from (7) we have

$(z + z)(x + y) = z(x + y) + z(x + y) = zx + zy + zx + zy$ for all $x , y ,z \in N$

On the other hand

$(z + z)(x + y) = (z + z)x + (z + z)y = zx + zx + zy + zy$ for all $x , y ,z \in N$.

By comparing both expressions, we obtain $zx + zy = zy + zx$ for all $x , y ,z \in N$ This is reduced to $z(x + y – x - y) = 0$ for all $x , y ,z \in N$, since N is left near ring ,then we get $(x + y - x - y) N(x + y - x - y) = \{0\}$, but N is semiprime near-ring so we conclude that $x + y - x - y = 0$ for all $x, y \in N$, then N is a ring .

**Corollary 3.2.** Let N be semiprime (prime) near – ring, if N admits an anti-homomorphism f, such that f is of period 2 on N then N is a ring .

**Theorem 3.4.** Let N be a 2-torsion free prime left near- ring, U is a nonzero semi group left ideal of N then N admits no nonzero α-derivation d such that $d\alpha = \alpha d$ and d is of period 2 on U.

**Proof**. Suppose that there exist nonzero α-derivation d on N such that $d^2(x) = x$ for all $x \in U$. For $x, y \in U$, $d(x)y \in U$ and the condition $d(x)y = d^2(d(x)y) = d(d^2(x)y + \alpha(d(x))d(y)) = d(xy + d(\alpha(x))d(y)) = d(xy) + d(d(\alpha(x))d(y)) = d(x)y + \alpha(x)d(y) + d^2(\alpha(x))d(y) + \alpha(d(\alpha(x))d^2(y) = d(x)y + 2\alpha(x)d(y) + \alpha^2(d(x))y$ we get

$2\alpha(x)d(y) + \alpha^2(d(x))y = 0$ for all $x, y \in U$ . (8)

$xy = d^2(xy) = d(d(xy)) = d(d(x)y + \alpha(x)d(y)) = d^2(x)y + \alpha(d(x))d(y) + d(\alpha(x))d(y) + \alpha^2(x)d^2(y) = xy + 2\alpha(d(x))d(y) + \alpha^2(x)y$, thus we get

$2\alpha(d(x))d(y) + \alpha^2(x)y = 0$ for all $x, y \in U$. (9)

Replacing x by rx in(9), where $r \in N$, and using Lemma 2.4 we get

$$\begin{aligned} 0 &= 2\alpha(d(rx))d(y) + \alpha^2(rx)y \\ &= 2d(\alpha(rx))d(y) + \alpha^2(r)\alpha^2(x)y \\ &= 2d(\alpha(r)\alpha(x))d(y) + \alpha^2(r)\alpha^2(x)y \\ &= 2d(\alpha(r))\alpha(x)d(y) + 2\alpha^2(r)d(\alpha(x))d(y) + \alpha^2(r)\alpha^2(x)y \\ &= 2d(\alpha(r))\alpha(x)d(y) + \alpha^2(r)(2d(\alpha(x))d(y) + \alpha^2(x)y). \end{aligned}$$

Using (9) in previous relation implies

$2d(\alpha(r))\alpha(x)d(y) = 0$ for all $x, y \in U$ and $r \in N$. (10)

substituting yr for r in (10), and using Lemma 2.4 we get

$0 = 2d(\alpha(yr)\alpha(x)d(y) = 2d(\alpha(y)\alpha(r))\alpha(x)d(y) = 2d(\alpha(y))\alpha(r)\alpha(x)d(y) + 2\alpha^2(y)d(\alpha(r))\alpha(x)d(y)$. Using (10) again implies $2d(\alpha(y))\alpha(r)\alpha(x)d(y) = 0$, since α is an automorphism, then we get $2d(\alpha(y))N\alpha(x)d(y) = 0$. 2-torsion freeness and primeness of N implies that either $d(\alpha(y)) = 0$ or $\alpha(x)d(y) = 0$ for all $x, y \in U$.

If $d(\alpha(y)) = 0$, i.e.; $\alpha(d(y)) = 0$ for all $y \in U$, since α is an automorphism then $d(y) = 0$ for all $y \in U$, hence $d(ry) = 0$ for all $y \in U$ and $r \in N$, i.e.; $d(r)y = 0$ for all $y \in U$ and $r \in N$ , By Lemma 2.2 we conclude $d = 0$. So we get a contradiction. Now if $\alpha(x)d(y) = 0$ for all $x, y \in U$. By (8) we obtain $\alpha^2(d(x))y = 0$ for all $x, y \in U$, so we have $\alpha^2(d(x))ry$

$= 0$ for all $x , y \in U$ and $r \in N$, that is $\alpha^2(d(x))N y = 0$ and primness of N ( similarly in the first case) implies either $d = 0$ or $U = 0$, so we have a contradiction.

**Corollary 3.3.** A 2- torsion free prime near – ring N, admits no α-derivation such that $d\alpha = \alpha d$ and of period 2 on N.

Now we prove some results about that what happens if we take α is an anti-homomorphism instead of automorphism in definitions of α-derivation and left α-centralizer.

**Theorem 3.5.** Let N be a prime near-ring. If N admits an additive endomorphism $d : N \longrightarrow N$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in N$ where α is an anti-homomorphism of period 2 then N is a commutative ring.

**Proof.** Since α is an anti-homomorphism of period 2, by Corollary 3.2 we conclude that N is a ring.

For all $x , y ,z \in N$, we have

$$d((xy)z) = d(xy)z + \alpha(xy)d(z) = (d(x)y + \alpha(x)d(y))z + \alpha(y)\alpha(x)d(z)$$

On the other hand

$d(x(yz)) = d(x)yz + \alpha(x)d(yz) = d(x)yz + \alpha(x)(d(y)z + \alpha(y)d(z)) = d(x)yz + \alpha(x)d(y)z+\alpha(x)\alpha(y)d(z)$, but $(d(x)y + \alpha(x)d(y))z = d(x)yz + \alpha(x)d(yz)$, so we get $\alpha(y)\alpha(x)d(z) = \alpha(x)\alpha(y)d(z)$, that is mean $\alpha(\alpha(y)\alpha(x)d(z)) = \alpha(\alpha(x)\alpha(y)d(z))$, we conclude $\alpha(d(z))xy = \alpha(d(z))yx$,

replacing x by xt in previous relation and using it again we have $\alpha(d(z))xty = \alpha(d(z))xyt$, so we have $\alpha(d(z))x(ty - yt) = 0$, that is $\alpha(d(z))N(ty - yt) = \{0\}$, primness of N yields either $\alpha(d(z)) = 0$ or $(ty - yt) = 0$, since α is bijective then we have $d = 0$ so we get a contradiction and this implies that $ty = yt$ for all $x , y \in N$. we conclude that N is a commutative ring .

**Theorem 3.6.** Let N be a prime near ring . Let f be a non- zero left α –centralizer , if f is of period 2 then α is of period 2 .

**Proof .** For all $x , y \in N$, we have $f(x)y = f^2(f(x)y) = f(f(f(x)y)) = f(f^2(x)\alpha(y)) = f(x\alpha(y)) = f(x)\alpha^2(y)$, i.e.; $f(x)(y - \alpha^2(y)) = 0$ for all $x , y \in N$ , replace x by tx, where $t \in N$, in previous relation we get $0 = f(tx)(y- \alpha^2(y)) = f(t)\alpha(x)(y- \alpha^2(y))$, since α is bijective we conclude that $f(t)N(y-\alpha^2(y)) = \{0\}$ for all $t, y \in N$, since $f \neq 0$, primness of N yields $y-\alpha^2(y) = 0$ for all $y \in N$, we conclude that α is of period 2 .

The converse of the previous Theorem is not true in general, the following example shows that α is of period 2 but f is not of period 2 .

Let S be a zero symmetric left near-ring . Suppose that

$$N=\left\{\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad : \quad x, y, 0 \in S\right\}.$$

Define $\alpha : N \longrightarrow N$, such that $\alpha = I$, where I is the identity mapping on N and define a map $f : N \longrightarrow N$ as follows

$$f\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

N is a zero symmetric left near-ring and f is an left $\alpha$-centralizer of N, $\alpha$ is of period 2 but f is not of period 2

In the following theorem, we investigate commutativity of prime near ring admitting a map $f : N \longrightarrow N$ such that $f(xy)=f(x)\alpha(y)$ if we take $\alpha$ as an anti-homomorphism of period 2 of N .

**Theorem 3.7.** Let N be prime near-ring. If N admits non-zero additive mapping $f : N \longrightarrow N$ such that $f(xy)=f(x)\alpha(y)$ for all x,y N, where $\alpha$ is an anti-homomorphism of period 2 on N, then N is a commutative ring.

**Proof**. Since $\alpha$ is an anti-homomorphism of period 2 then by Corollary 3.2 we conclude that N is a ring.

$f(xy)=f(x)\alpha(y)$ for all $x,y \in N$. (11)

replace y by yt in (11), where $t \in N$, we get

$f(xyt) = f(x)\alpha(yt) = f(x)\alpha(t)\alpha(y)$

On the other hand we have

$f(xyt) = f(xy)\alpha(t) = f(x)\alpha(y)\alpha(t)$, hence $f(x)\alpha(y)\alpha(t) = f(x)\alpha(t)\alpha(y)$ for all $x,y,t \in N$. So we get $f(x)(\alpha(y)\alpha(t)- \alpha(t)\alpha(y)) = 0$for all $x,y,t \in N$, since $\alpha$ is bijective then we can replace $\alpha(y)$ by r and $\alpha(t)$ by s, where $r, s \in N$. So we have

$f(x)(rs-sr) = 0$ for all $r, s \in N$. (12)

Replace x by xn, where $n \in N$, in (12) we get $f(xn)(rs - sr) = 0$, that is

$f(x)\alpha(n)(rs - sr) = 0$, hence $f(x)N(rs - sr) = \{0\}$, since N is a prime and $f \neq 0$ then $rs = sr$ for all $r,s \in N$.i.e.; N is commutative ring.

The following example shows that the primness of N is a necessary condition in the previous theorem

Let S be a zero symmetric left near-ring . Suppose that

$$N=\left\{\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad : \quad x, y, 0 \in S\right\}.$$

Define a map α :N ⟶N as follows

$$\alpha\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & y & x\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Define f :N ⟶N such that

$$f\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

N is a zero symmetric left near-ring and f is an additive mapping satisfying f(xy)=f(x)α(y) for all x,y ϵ N, α is an anti-homomorphism of period 2, however N is not a ring.

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## الدوال ذات الدورة 2 على الحلقات المقتربة

**Abstract**

In this paper , we introduce the notion of maps of period 2 on near-ring N , the main our purpose is to study and investigate the existence and properties of mapping such as homomorphisms, anti - homomorphisms, $\alpha$ -derivations and $\alpha$-centeralizers when they are of period 2 on near – rings.

**المستخلص:**

**قدمنا في هذا البحث تعريفا للدالة ذات الدورة 2 على الحلقات المقتربة وكان الغرض الرئيسي من هذا البحث هو دراسة وجود وخواص بعض الدوال مثل التشاكلات و تشاكلات ضد واشتقاقات $\alpha$ وتمركزات من النوع $\alpha$ عندما تكون هذه الدوال ذات الدورة 2 على الحلقات المقتربة .**