On Derivations of Period 2 On Near–Rings

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Abstract

In this paper , we introduce the notion of mapping of period 2 on near-ring N. Also we investigate the existence and properties of derivations and generalized derivations of period 2 on near – rings.

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1. Introduction

A right near – ring (resp.left near ring) is a set N together with two binary operations (+) and (.) such that (i) (N,+) is a group (not necessarily abelian). (ii) (N,.) is a semi group. (iii) For all a,b,c ϵ N ; we have (a+b).c = a.c + b.c (resp. a.(b+c) = a.b + b.c. Through this paper, N will be a zero symmetric left near – ring (i.e., a left near-ring N satisfying the property 0.x=0 for all x$\in$ N). we will denote the product of any two elements x and y in N ,i.e.; x.y by xy. The symbol Z will denote the multiplicative centre of N, that is Z=$\{x \in N \mid xy = yx \; for \; all \; y \in N\}$. N is called a prime near-ring if xNy = {0} implies either x = 0 or y = 0. It is called semiprime if xNx={0} implies x=0. Near-ring N is called n-torsion free if nx = 0 implies x=0. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal) if NU$\subseteq$U (resp.UN$\subseteq$U), If U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. A normal subgroup (I,+) of (N, +) is called a right ideal (resp. left ideal) of N if (x + i)y – xy $\in$ I for all x,y $\in$ N and i $\in$ I (resp. xi $\in$ I for all i$\in$I and x$\in$N). I is called ideal of N if it is both a left ideal as well as a right ideal of N. For terminologies concerning near-rings, we refer to Pilz [6].

The concept of mapping of period 2 has already introduced in ring by Bell . H.E and Daif. M. N [2], in the present paper, motivated by this concept we define a mapping of period 2 on a near-ring N. A mapping of the near ring N into itself is of period 2 on N if $f^2(x) = x$ for all x$\in$ N. Let U be a non empty subset of N, a mapping f:N → N is of period 2 on U if $f^2(x) = x$ for all x$\in$ U**.**

An additive mapping d:N $\longrightarrow$ N is said to be derivation of N if d(xy) = xd(y) + d(x)y , or equivalently, as noted in ([1], lemma 4) that d(xy) = d(x)y + xd(y) for all x,y ϵ N. The concept of derivation has been generalized in several ways by various authors. the notion of generalized derivation has been already introduced and study in near-rings by

Öznur Gölbasi [5]**.** An additive mapping $f:N \rightarrow N$ is called a right generalized derivation with associated derivation d if $f(xy) = f(x)y + xd(y)$, for all $x,y \in N$ and f is called a left generalized derivation with associated derivation d if $f(xy) = d(x)y + xf(y)$, for all $x,y \in N$. f is called a generalized derivation with associated derivation d if it is both a right as well as a left generalized derivation with associated derivation d.

Many authors studied the relationship between structure of near – ring N and the behaviour of special mapping on N (see [1],[3],[4],[5],[7] for reference where further references can be found). In the year 2014 Bell. H. E and Daif. M. N. in [2] studied the existence and properties of derivations and generalized derivations of period 2 on prime and semiprime rings. Motivated by these works, we have extended these results in the setting of derivations and generalized derivations of period 2 on certain subset of prime and semiprime near-ring.

2. Preliminaries

The following lemmas are essential for developing the proofs of our main results. .

Lemma 2.1.[7] Let N be near-ring and d be arbitrary derivation of N, then

(i) $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$;

(ii) $(d(x)y + xd(y))z = d(x)yz + xd(y)z$; for all $x , y , z \in N$.

Lemma 2.2.[5] Let N be prime near-ring admitting a derivation d and $a \in N$, if $ad(N) = 0$, then $a =0$.

Lemma 2.3.[1] Let N be a prime near-ring. If U is non-zero semigroup right ideal (resp, semigroup left ideal) and x is an element of N such that $Ux = \{0\}$(resp, $xU = \{0\}$), then $x = 0$.

Lemma 2.4.[5] Let N be a 2–torsion free prime near-ring and d is derivation of N if $d^2= 0$, then $d = 0$.

Lemma 2.5.[7] Let N be a near-ring**.** If N admits a derivation d, then $d(Z)\subseteq Z$.

Lemma 2.6.[5] Let N be a near-ring

(i) Let f be a right generalized derivation of N with associated derivation d. Then $f(xy) = xd(y) + f(x)y$ for all $x , y \in N$.

(ii) Let f be a left generalized derivation of N with associated derivation d. Then $f(xy) = xf(y) + d(x)y$ for all $x , y \in N$.

Lemma 2.7.[5] Let N be a near-ring**.**

(i) Let f be a right generalized derivation of N with associated derivation d. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x,y,z \in N$.

(ii) Let f be a left generalized derivation of N with associated derivation d. Then $(d(x)y + xf(y))z = f(x)yz + xd(y)z$ for all $x,y,z \in N$.

Lemma 2.8. Let N be near–ring admitting a derivation d of N then

$$2(d(x)y + xd(y))z = 2d(x)yz + 2xd(y)z \, .$$

Proof. By Lemma 2.1(ii) we have

$$\begin{aligned} 2(d(x)y + xd(y))z &= 2(d(x)yz + xd(y)z) \\ &= d(x)yz + xd(y)z + d(x)yz + xd(y)z \\ &= d(x)yz + (xd(y) + d(x)y)z + xd(y)z \\ &= d(x)yz + (d(x)y + xd(y))z + xd(y)z \\ &= d(x)yz + d(x)yz + xd(y)z + xd(y)z \\ &= 2d(x)yz + 2xd(y)z \, . \end{aligned}$$

Lemma 2.9. Let N be near – ring admitting a generalized derivation f with associated derivation d then $f(Z) \subseteq Z$.

Proof. Let $z \in Z$, $x \in N$, then $f(zx) = f(xz)$; hence $zd(x) + f(z)x = d(x)z + xf(z)$, therefore $f(z)x = xf(z)$ for all $x \in N$, we conclude that $f(Z) \subseteq Z$.

3. Main Result

In this section we study derivation and generalized derivation when they are of period two.

If we consider f is a derivation of period 2, its clear that f is bijective , so there exists no $x \neq 0$ such that $d(x) = 0$, it follows that the near-ring with 1 admits no derivation which is of period 2 on N. We will show that a semiprime near-ring N admits no derivation of period 2 on N.

Theorem 3.1. Let N be semiprime near ring and U is anon zero semigroup left ideal of N. Then N admits no derivation such that d is of period 2 on U.

Proof. Suppose that there exist derivation d on N such that $d^2(x) = x$ for all $x \in U$. For all $x , y \in U$, $d(x)y \in U$ and the condition $d(x)y = d^2(d(x)y) = d(d^2(x)y + d(x)d(y)) = d(xy + d(x)d(y)) = d(xy) + d(d(x)d(y)) = d(x)y + xd(y) + d^2(x)d(y) + d(x)d^2(y) = d(x)y + 2xd(y) + d(x)y$ yeilds

$$d(x)y + 2xd(y) = 0 \quad \text{for all } x , y \in U. \qquad (1)$$

Since $xy = d^2(xy) = d(d(xy)) = d(d(x)y + xd(y)) = d^2(x)y + d(x)d(y) + d(x)d(y) + xd^2(y) = xy + d(x)d(y) + d(x)d(y) + xy$, so we get

$$xy + 2d(x)d(y) = 0 \qquad \text{for all } x , y \in U \qquad (2)$$

Replacing x by rx, where $r \in N$, in (2) we obtain $0 = rxy + 2d(rx)d(y) = rxy + 2(rd(x) + d(r)x)d(y)$. Using Lemma 2.8 in previous relation we get

$rxy + 2rd(x)d(y) + 2d(r)xd(y) = 0$. i.e.;

$0 = rxy + rd(x)d(y) + rd(x)d(y) + 2d(r)xd(y) = r(xy + 2d(x)d(y)) + 2d(r)xd(y)$

Using (2) in previous relation implies

$$2d(r)xd(y) = 0 \text{ for all } x, y \in U \text{ and } r \in N. \quad (3)$$

Substituting yr for r in (3), we get $0 = 2d(yr)xd(y) = 2(d(y)r + yd(r))xd(y)$, by lemma 2.8 we get $2d(y)rxd(y) + 2yd(r)xd(y) = 0$, using (3) in previous relation implies $2d(y)rxd(y) = 0$, hence $2xd(y)rxd(y) = 0$. Semiprimeness of N yields $2xd(y) = 0$ for all $x, y \in U$, then relation (1) can be reduced to $d(x)y = 0$ for all $x, y \in U$. Therefore $d(d(x)y) = d^2(x)y + d(x)d(y) = xy + d(x)d(y) = 0$ for all $x, y \in U$. Which together with (2) yields $xy = 0$ for all $x, y \in U$, therefore $xU=0$ for all $x \in U$, by Lemma 2.3 we obtain $x = 0$ for all $x \in U$, so we get a contradiction.

Corollary 3.1. A semiprime near – ring N admits no derivation of period 2 on N.

Any near- ring admits a right generalized derivation of period 2 namely, the identity map and its negative. Moreover, if N has 1 and $c \in N$ such that $c^2=1$ then $f(x) = cx$ define a right generalized derivation of period 2 on N. Now we show that in many prime near-rings there are no other possibilities.

Theorem 3.2. Let N be a 2-torsion free prime near-ring. Let f be a right generalized derivation on N with associated derivation d. If f is of period 2, then $d(Z) = \{0\}$.

Proof. For all $x,y \in N$, we have

$xy = f^2(xy) = f(f(x)y + xd(y)) = f(f(x)y) + f(xd(y)) = f^2(x)y + f(x)d(y) + f(x)d(y) + xd^2(y) = xy + 2f(x)d(y) + xd^2(y)$. i.e.;

$$2f(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in N \quad (4)$$

Replacing x by f(x) in (4) yields

$$2xd(y) + f(x)d^2(y) = 0 \text{ for all } x, y \in N \quad (5)$$

Letting $z \in Z$ and $x \in N$ and replacing x by xz in (5), we get $2xzd(y) + f(xz)d^2(y) = 0$, by Lemma 2.7, proceeding relation takes the form

$2xzd(y) + f(x)zd^2(y) + xd(z)d^2(y) = 0$. But $z \in Z$, hence we get

$0 = 2zxd(y) + zf(x)d^2(y) + xd(z)d^2(y) = z(2xd(y) + f(x)d^2(y)) + xd(z)d^2(y)$, using (5) in previous relation we get $x\, d(z)d^2(y) = 0$, by Lemma 2.5 we conclude that $d(z)xd^2(y) = 0$ for all $x, y \in N$ and $z \in Z$. i.e.; $d(z)\, N\, d^2(y) = \{0\}$, primness of N yields that either $d(z) = 0$ or $d^2(y) = 0$, if $d(z) \neq 0$ then $d^2(y) = 0$ then by Lemma 2.4 we conclude that $d = 0$. Thus $d(Z) = \{0\}$.

Theorem 3.3. let N be a 2-torsion free prime near-ring with 1. If a right generalized derivation f associated with d given by $f(x) = x + d(x)$ (resp $f(x) = -x + d(x)$) for all $x \in N$, of period 2 on N then f is the identity map (resp , the negative of the identity map) on N.

Proof. Consider the case $f(x) = x + d(x)$ for all $x \in N$. If f is of period 2, we have

$x = f^2(x) = f(f(x) = f(x + d(x)) = x + d(x) + d(x) + d^2(x)$. i.e.;

$$2d(x) + d^2(x) = 0 \text{ for all } x \in N \qquad (6)$$

replacing x by xy in(6), we get

$$\begin{aligned} 0 &= 2d(xy) + d^2(xy) \\ &= 2(d(x)y + xd(y)) + d(xd(y) + d(x)y) \\ &= 2d(x)y + 2xd(y) + xd^2(y) + d(x)d(y) + d(x)d(y) + d^2(x)y \\ &= 2d(x)y + x(2d(y) + d^2(y)) + 2d(x)d(y) + d^2(x)y \end{aligned}$$

for all $x , y \in N$. Using (6) in previous relation implies

$$2d(x)y + 2d(x)d(y) + d^2(x)y = 0 \qquad (7)$$

since $1 \in N$, hence $1 \in Z$ and by Theorem 2.3 we conclude that $d(1) = 0$, by lemma 2.7(i) we have $(f(x)y + xd(y)z = f(x)yz + xd(y)z$ (8)

Take $x = 1$ in (8) we get $(f(1)y + d(y))z = f(1)yz + d(y)z$; hence we get the partial distributive law $(y + d(y))z = yz + d(y)z$ (9)

replacing y by d(y) in (9), we obtain

$$(d(y) + d^2(y))z = d(y)z + d^2(y)z \qquad (10)$$

Frome relation (7) we obtain $d(x)y + 2d(x)d(y) = -d(x)y - d^2(x)y$, using (10) in previous relation implies $d(x)y + 2d(x)d(y) = -(d(x) + d^2(x))y$, using (6) in previous relation, we get $d(x)y + 2d(x)d(y) = d(x)y$. i.e.; $2d(x)d(y) = 0$ for all $x , y \in N$, since N is 2-torsion free then $d(x)d(y) = 0$ for all $x , y \in N$ **,** then $d(x) d(N) = 0$ for all $x \in N$. By lemma 2.2, we conclude that $d(x) = 0$ for all $x \in N$. Thus f is the identity map on N

A similar argument works if $f(x) = -x + d(x)$ $\forall$ $x \in N$.

Theorem 3.4 Let N be a 2-torsion free prime near-ring with 1. N has no nonzero divisors. If f is a right generalized derivation on N of period 2, then f is the identity map or its negative.

Proof . Note that $f(x) = f(1.x) = f(1)x + d(x)$ for all $x \in N$. (11)

For all $x,y \in N$, we have

$xy = f^2(xy) = f(f(x)y + xd(y)) = f(f(x)y) + f(xd(y)) = f^2(x)y + f(x)d(y) + f(x)d(y) + xd^2(y) = xy + 2f(x)d(y) + xd^2(y)$. i.e.;

$$2f(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in N \tag{12}$$

Replacing x by f(x) in (4) yields

$$2xd(y) + f(x)d^2(y) = 0 \text{ for all } x , y \in N \tag{13}$$

Taking x = 1 in (12) and (13). we have

$2f(1)d(y) + d^2(y) = 0$ and $2d(y)+f(1)\, d^2(y)=0$ for all $x \in N$. It follows that $2d(y) - 2d(y)f(1)^2 = 0$, that is $2d(y)(1 - f(1)^2) = 0$, since N is 2-torsion free then $d(y)(1 - f(1)^2) = 0$. But N has no nonzero divisor if $d \neq 0$, we obtain $0 =1 - f(1)^2 = 1-f(1)+f(1) - f(1)^2 = (1-f(1)) + f(1)(1-f(1)) = (1-f(1)) + (1-f(1))f(1) = (1-f(1))(1+f(1))$, since N has no nonzero divisor then we get either f(1) = 1 or f(1) = -1, from (11) we have f(x) = x + d(x) or f(x) = -x + d(x). So by Theorem 3.3 this would imply d = 0, thus we get

$$f(x) = f(1)x \quad \text{for all } x \in N. \tag{14}$$

Since f is of period 2 then $x = f^2(1)x = x\, f^2(1)$ and $x(1- f^2(1)) = 0$ for all $x \in N$. Thus we conclude that f(1) = 1 or f(1) = -1, by (12) we find that f is the identity map or its negative .

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الاشتقاقات ذات الدورة 2 على الحلقات المقتربة

Abstract

In this paper , we introduce the notion of mapping of period 2 on near-ring N and investigate the existence and properities of derivations and generalized derivations of period 2 on near – rings.

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المستخلص :

قدمنا في هذا البحث تعريفا للدوال ذات الدورة 2 على الحلقات المقتربة وقمنا بدراسة وجود وخواص الاشتقاقات والاشتقاقات المعممة عندما تكون من الدوال ذات الدورة 2 .