Korovkin Type Theorem in Weighted Spaces ($L_{p,\alpha}$-Space)

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Abstract:

In this study, we obtain some Korovkin type approximation theorem by positive linear operators on $L_{p,\alpha}$-space of unbounded functions by using the test functions $\{1,x,x^2\}$.

Introduction:

The approximation problem arises in many contexts of "Numerical Analysis and Computing". The "Quadrature" is one such context, for example. Wierstrass (1885) proved his celebrated approximation theorem: If $f \in C[a,b]$; for $\delta > 0$ there is a polynomial p such that $|f - p|<\delta$. In other words, the result established the existence of an algebraic polynomial in the relevant variable.

The great Russian mathematician S. N. Bernstein proved the Weirstrass theorem in such a manner as was very stimulating and interesting in many ways.

As we know, approximation theory has important applications in various areas of mathematics (see [3,4,7]). Most of the classical operators tend to converge to the value of the function being approximated.

It is also known J. P. King [6] has presented an unexpected example of operators of Bernstein type which preserve that test function e_0 and e_2 of Bohman-Korovkin theorem.

Now, we constructive approximation of unbounded functions over L_n which is a linear positive operators from $B_\alpha = \{f : A \longrightarrow R$ such that $|f(x)e^{-\alpha x}| \le M\}$

into itself, we use the special norm:

$$||f||_{p,\alpha} = \left(\int_A \left| f(x)e^{-\alpha x} \right|^p dx \right)^{1/p} < \infty,\ \alpha > 0$$

Here, we define the weighting modulus of $f \in B_\alpha$ by:

$$\omega(f,\delta)_{p,\alpha} = \left(\int_A \left| \overset{1}{\underset{h}{\Delta}} f(x)e^{-\alpha x} \right|^p dx \right)^{1/p}$$

where $\overset{1}{\underset{h}{\Delta}} f(x) = |f(x) - f(y)|,\ |x - y| < h$ for all $h < \delta$.

Main Results:

By using Korovkin theorem in[7],[5],[3], we want to estimate the degree of best approximation of unbounded function in terms of weighted modulus of function.

<u>Theorem:</u>

Let f be unbounded real-valued function on [a,b], let $L_n : B_\alpha \longrightarrow B_\alpha$ be a linear positive operator, such that:

$$||L_n(g) - g||_{p,\alpha} \longrightarrow 0, \text{ for } n \longrightarrow \infty \text{ and } g \in \{1,x,x^2\}$$

Then:

$$||L_n(f) - f||_{p,\alpha} \longrightarrow 0, \text{ for } n \longrightarrow \infty, \text{ for each } f \in B_\alpha$$

Proof:

From [5] , $||L_n(g) - g||_{p,\alpha} \longrightarrow 0$ for $n \longrightarrow \infty$, we get:

$$L_n(g(y))(x) = g(x) + \omega\big((g(y))(x), \delta \big)_{p,\alpha},$$

where $\omega\left((g(y))(x),\delta \right)$ represents the error we make in approximating g(x) with $L_n(g(y))(x)$, and hence it is such that:

$$\|\omega((g(y))(x),\delta)\|_{p,\alpha} \longrightarrow 0$$

Now, we have to prove that:

$$\|L_n(f) - f\|_{p,\alpha} \longrightarrow 0, \text{ for } n \longrightarrow \infty, \ \forall\ f \in B_\alpha$$

we fixed $f \in B_\alpha$ and $\omega(f,\delta)_{p,\alpha} > 0$, we want to prove that there exist $\bar{n}$ (as a function of f and ω), such that for any $n \geq \bar{n}$ and for every $x \in R$, we have:

$$|(L_n(f) - f)e^{-\alpha x}| < \omega(f,\delta)_{p,\alpha}.$$

And we write:

$$\underset{n}{\Delta}(f)(x) = \left|(L_n(f(y))(x) - f(x))e^{-\alpha x}\right|$$

$$= \left|(L_n(f(y))(x) - 1.f(x))e^{-\alpha x}\right|$$

$$= \left|(L_n(f(y))(x) - [L_n(1)(x) - \omega(f,\delta)_{p,\alpha}(x)]f(x)e^{-\alpha x}\right|$$

since $1 = L_n(1)(x) - \omega(f,\delta)_{p,\alpha}(x)$

$$\underset{n}{\Delta}(f)(x) \leq \left|(L_n(f(y))(x) - L_n(1)f(x))e^{-\alpha x}\right| + |f(x)|\ |\omega(f,\delta)_{p,\alpha}|e^{-\alpha x}$$

$$\leq \left|(L_n(f(y))(x) - L_n(1)f(x))e^{-\alpha x}\right| + \left|f(x)\ e^{-\alpha x}\right|\ |\omega(f,\delta)_{p,\alpha}|$$

$\leq \left|(L_n(f(y))(x) - L_n(1)f(x))e^{-\alpha x}\right| + \|f(x)\|\ |\omega(f,\delta)_{p,\alpha}|$ and from[5] and [2] we have

$$\leq \left|(L_n(f(y))(x) - L_n(1)f(x))e^{-\alpha x}\right| + C_p\left(\int_A \left|f(x)e^{-\alpha x}\right|^p M\,\omega(f,\delta)_\alpha(x)\,dx\right)^{1/p}$$

$$= \left|(L_n(f(y))(x) - L_n(1)f(x))e^{-\alpha x}\right| + \|f\|_{p,\alpha}\ \|w(f,\delta)_\alpha\|_p$$

we have to find:

$$\underset{n}{\Delta}(f)(x) \leq L_n\left(|f(y) - f(x)|\, e^{-\alpha x} \right)(x)\ |f(y) - f(x)|e^{-\alpha x}(x) + \frac{\omega}{4},\ \forall\ n \geq \bar{n}$$

Whenever $\|w(f,\delta)_\alpha\|_p \longrightarrow 0$ for $n \longrightarrow \infty$. Now:

$$\left(\int_A \left| f(y)e^{-\alpha y} - f(x)e^{-\alpha x} \right|^p dx \right)^{1/p} = \| \underset{n}{\Delta} f \|_{p,\alpha}$$

$$= \omega(f,\delta)_{p,\alpha}, \text{ where } \delta = |x - y|$$

and $\omega(f,\delta)_{p,\alpha} \longrightarrow 0$. If $|x - y| \longrightarrow 0$, then from [1] we have:

$$|f(x)e^{-\alpha x} - f(y)e^{-\alpha y}| < \bar{\omega}\,\chi\{z : |z - x| < \delta\}(y) + 2\|f\|_{p,\alpha}\chi\{z : |z - x| \geq \delta\}(y)\ ,$$

Now, $|z - x| \geq \delta \Leftrightarrow \dfrac{|z-x|}{\delta} \geq 1 \Rightarrow \dfrac{(z-x)^2}{\delta^2} \geq 1$

In conclusion, in the set the inequality is satisfied, we can write:

$$\chi\{x : |z - x| \geq \delta\}(y) = 1 \leq \frac{(y-x)^2}{\delta^2}$$

otherwise, we have:

$$\chi\{x : |z - x| \geq \delta\}(y) = 0 \leq \frac{(y-x)^2}{\delta^2}$$

Therefore, uniformly with respect to $x, y \in R$ for any $\bar{\omega} > 0$, we find $\delta = \delta\bar{\omega}$, such that:

$$|f(x)e^{-\alpha x} - f(y)e^{-\alpha y}| \leq \bar{\omega}(1) + 2\|f\|_{p,\alpha}\frac{(y-x)^2}{\delta^2}$$

Choose $\bar{\omega} = \dfrac{\omega}{4}$.

By applying the operator $L_n(.)$ for every $n \geq \bar{n}_1$, we find that:

$$\underset{n}{\Delta}(f)(x) \leq \frac{\omega}{4} + L_n\left(\bar{\omega} + \frac{2\|f\|_{p,\alpha}}{\delta^2}(y-x)^2 \right)(x)$$

$$= \frac{\omega}{4} + \bar{\omega}\,L_n(1) + \frac{2\|f\|_{p,\alpha}}{\delta^2} L_n\left(y^2 - 2yx + x^2 \right)(x)$$

Hence, $L_n(1)(x) = \left(1 + \omega(1,\delta)_{p,\alpha}\right)(x)$ and the constant 1 is a test function.

Therefore ,from[3],[5] we have:

$$\underset{n}{\Delta}(f)(x) \le \frac{\omega}{4} + \bar{\omega}\left(1 + \omega(1,\delta)_{p,\alpha}(x)\right) + \frac{2\|f\|_{p,\alpha}}{\delta^2}\left\{x^2 + \omega_\alpha(y^2,\delta)(x) - 2x\left[x + \omega_\alpha(y,\delta)(x)\right] + x^2\left(1 + \omega_\alpha(1,\delta)(x)\right)\right\}$$

Now, by definition of limit, there exist a functions $\bar{n}_2$, $\bar{n}_3$ such that for $n \ge \bar{n}_2$, we have

$\omega_\alpha(1,\delta) \le 1$, uniformly with respect to $x \in R$ and for $n \ge \bar{n}_3$, we have:

$$\|f\|_{p,\alpha}\,\|\omega_\alpha(y^2,\delta)(x) - 2x\omega_\alpha(y,\delta)(x) + x^2\omega_\alpha(1,\delta)(x)\|_{p,\alpha} \le \frac{\delta^2\bar{\omega}}{2}$$

Now, by taking $n \ge \bar{n}$, with $\bar{n} = \max\{\bar{n}_1, \bar{n}_2, \bar{n}_3\}$ (depending on both f and ω), we have:

$$\underset{n}{\Delta}(f)(x) \le \frac{\omega}{4} + 2\bar{\omega} + \frac{2\|f\|_{p,\alpha}}{\delta^2}\{\omega_\alpha(y^2,\delta)(x) - 2x\omega_\alpha(y,\delta)(x) + x^2\omega_\alpha(1,\delta)(x)\}$$

$$\le \frac{\omega}{4} + 3\bar{\omega} = 1$$

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نوع معين من نظرية كورفكين في الفضاء الوزني ($L_{p,\alpha}$-Space)

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المستخلص:

في هذه الدراسة إستنتجنا نوع من تقريب كورفكين بواسطة المؤثرات الخطية الموجبة للدوال غير المقيدة في الفضاء الوزني ($L_{p,\alpha}$-Space).