On a Q- Smarandache Implicative Ideal with Respect to an Element of a Q-Smarandache BH-algebra

المثالية Q- سمرندش الاستناجية بالنسبة الى عنصر في جبر -BH سمرندش-Q

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Abstract

In this paper, we define the concept of a Q-Smarandache implicative ideal with respect to an element of a Q-Smarandache BH-algebra. We state and prove some theorems which determine the relationships among this notion and other types of ideals of a Q-Smarandache BH-algebra.

الخلاصة

عرفنا في هذا البحث مفهوم(المثالية -Q سمرندش الاستناجية بالنسبه لعنصر في جبر-BH سمرندش-Q , وأعطينا وبرهنا بعض المبرهنات التي تحدد العلاقة بين هذا المفهوم مع بعض أنواع المثاليات في جبر -BH سمرندش-Q).

1. INTRODUCTION

The notion of BCK-algebra and BCI-algebra was formulated first in 1966 by Y.Imai and K.Iseki as a generalization of the concept of set-theoretic difference and propositional calculus [4]. In 1983, Q.P.Hu and X.Li introduced the ntion of BCH-algebra which are generalization of BCK\BCI -algebra [5]. In 1998, Y. B. Jun, E. H. Roh and H .S. Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebra [7]. In 2009, A.B.Saeid and A.Namdar introduced the notion of a Q-Smarandache BCH-algebra and Q-Smarandache ideal of a Q-Smarandache BCH-algebra, these notion were generalized to BH-algebra in 2012 by H.H.Abass and S.J.Mohammed[2]. In 2014, H. H. Abbass and S. A. Neamah introduced the notion of an implicative ideal with respect to an element of a BH-algebra[1]. In this paper, a new type of a Q-Smarandache ideal of Q-Smarandache BH- algebra, namely a Q-Smarandache implicative ideal with respect to an element is introduced some related properties investigated .

2. PRELIMINARIES

In this section, we review some basic concepts about a BCK-algebra, BCI- algebra,BCH-algebra, BH-algebra, Smarandache BH-algebra, (ideal, positive implicative and implicative ideal with respect to an element) of a BH-algebra and Q-Smarandach ideal of a Q-Smaradache BH-algebra, with some theorems and propositions .

Definition (2.1) :[8]

A BCI-algebra is an algebra (X,*,0), where X is a nonempty set, "*" is a binary operation and 0 is a constant, satisfying the following axioms:

i. $((x*y)*(x*z))*(z*y) = 0$, for all $x, y, z\in X$.
ii. $(x*(x*y))*y = 0$, for all $x, y\in X$.
iii. $x * x = 0$, for all $x\in X$.
iv. $x * y =0$ and $y * x = 0$ imply $x = y$, for all $x, y\in X$.

Definition (2.2) :[4]

A BCK-algebra is a BCI-algebra satisfying the axiom: $0 * x = 0$, for all $x \in X$.

Definition(2.3):[5]

A **BCH-algebra** is an algebra $(X,*,0)$, where X is a nonempty set,"*" is a binary operation and 0 is a constant, satisfying the following axioms:
i. $x * x = 0, \forall x\in X$.
ii. $x * y =0$ and $y * x = 0$ imply $x = y, \forall x, y\in X$.
iii. $(x * y) * z = (x * z) * y, \forall x, y, z\in X$.

Definition (2.4) :[7]

A **BH-algebra** is a nonempty set X with a constant 0 and a binary operation"*" satisfying the following conditions:
i. $x*x=0, \forall x\in X$.
ii. $x*y=0$ and $y*x =0$ imply $x = y, \forall x, y\in X$.
iii. $x*0 =x, \forall x\in X$.

Definition (2.5) :[3]

A **bounded BCK-algebra** satisfying the identity $x * (y * x) = x, \forall x, y \in X$.

Definition (2.6) :[7]

Let I be a nonempty subset of a BH-algebra X. Then I is called an **ideal** of X if it satisfies:
i. $0\in I$.
ii. $x*y\in I$ and $y\in I$ imply $x\in I$.

Definition (2.7):[1]

A nonempty subset I of a BH-algebra X is called **an implicative ideal with respect to an element b of a BH-Algebra** (or briefly **b-implicative ideal**), $b\in X$. if
i. $0\in I$.
ii. $((x*(y*x))*z)*b \in I$ and $z \in I$ imply $x\in I, \forall x, y, z\in X$.

Definition(2.8):[6]

A BH-algebra $(X,*,0)$ is said to be **a positive implicative** if it satisfies for all x,y and $z\in X$, $(x*z)*(y*z)=(x*y)*z$.

Definition(2.9):[2]

A Smarandache BH-algebra is defined to be a BH-algebra X in which there exists a proper subset Q of X such that :
i. $0\in Q$ and $|Q|\geq 2$.
ii. Q is a BCK-algebra under the operation of X.

Definition(2.10):[2]

Let X be a Smarandache BH-algebra**.** A nonempty subset I of X is called a **Smarandache ideal of X related to Q** (or briefly**, Q-Smarandache ideal** of X) if it satisfies:
i. $0 \in I$.
ii. $\forall y \in I$ and $x*y \in I\Rightarrow x \in I, \forall x\in Q$.

Proposition(2.11) :[2]

Let{I_i, i∈λ} be a family of Q-Smarandache ideals of a Smarandache BH-algebra X. Then $\bigcap_{i\in\lambda} I_i$ is a Q-Smarandache ideal of X.

Proposition(2.12): [2]

Let{I_i, i∈λ} be a chain of a Q-Smarandache ideals of a Smarandache BH-algebra X. Then $\bigcup_{i\in\lambda} I_i$ is a Q-Smarandache ideal of X.

Proposition (2.13) :[2]

Let X be a Smarandache BH-algebra . Then every ideal of X is a Q-Smarandache ideal of X.

Theorem (2.14):[2]

Let Q_1 and Q_2 be BCK-algebras contained in a Smarandache BH- algebra X and $Q_1 \subseteq Q_2$. Then every Smarandache ideal of X related to Q_2 is a Smarandache ideal of X related to Q_1.

3. THE MAIN RESULTS

In this section, we introduce the concept of a **Q-Smarandache implicative ideal** of a Q-Smarandache BH-algebra. Also, we state and prove some theorems and examples about these concepts.

Definition (3.1):

Let I be a Q-Smarandach ideal of a Q- Smarandache BH-algebra X and b ∈ X. Then I is called a **Q-Smarandache implicative ideal with respect to b** (denoted by a **Q - Smarandache b-implicative ideal**) if :

((x*(y*x))*z)*b ∈ I and z ∈ I imply x∈ I, ∀ x,y ∈Q.

Example (3.2):

Consider the Q-Saramdache BH-algebra X= {0, 1, 2, 3} with the binary operation " * " defined by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	2	0

where Q = {0,2} is a BCK- algebra .

The Q-Smarandache ideal I = {0,1} is a Q-Smarandache 0-implicative ideal of X, so I be a Q-Smarandache 1,3 - implicative ideal of X, but it is not a Q- Smarandache 2-implicative ideal of X. Since, x=2, y=2, z= 0, ((2*(2*2))*0)*2=((2*0)*2 =2*2=0 ∈I ,but x=2∉I .

Proposition (3.3) :

Let X be a Q-Smarandache BH-algebra. Then every b- implicative ideal of X is a Q-Smarandache b-implicative ideal of X, ∀ b ∈ X.

Proof:

Let I is b - implicative ideal of X, ∀ b ∈ X.
Now, let x,y ∈ Q and z ∈ I such that ((x*(y*x)) * z)* b ∈ I and z ∈ I.
Since x,y ∈ Q ⟹ x,y ∈ X. [Since Q⊆ X]
Now, we have
((x*(y*x))* z)*b ∈ I and z ∈ I.
⟹ x ∈ I. [Since I is b- implicative ideal of X, by Definition (2.7) (ii)]
Therefore, I is a Q-Smarandache b- implicative ideal of X. ■

Remark (3.4) :

The follwing example shows that converse of Proposition(3.3) is not correct in general .

Example (3.5) :

Consider the Q-Smarandache BH-algebra X={0,1,2,3}with binary operation "*" defind by the following table:

*	0	1	2	3
0	0	0	2	3
1	1	0	1	2
2	2	1	0	1
3	3	3	2	0

where Q={0,1} a is BCK - algebra.
The Q-Smarandache ideal I={0,2} is a Q-Smarandache 2- implicative ideal of X, but it is not an 2 - implicative ideal of BH- algebra. Since, x=3, y=0, z =2, ((3*(0*3))*2)*2= ((3*3)*2)*2 = (0*2)*2 = 2*2= 0 ∈ I, but 3 ∉ I .

Theorem (3.6) :

Let (N,*) be a Q-Smarandache BH-algebra, where N={0,1,2,….}, " * " be a binary operation defind on N by **:**

$$x*y=\begin{cases} x & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}, \forall\, x, y \in N$$

,and Q={4k , k ∈ N} is a BCK- algebra. Then I= {2k , k ∈ N} is a Q-Smarandache b- implicative ideal of N, ∀ b ∈ I .

Proof:

It is clear I is a Q-Smarandache ideal of N.
Now, let x,y ∈ Q and z, b ∈ I such that ((x*(y*x)* z) *b ∈ I and z ∈ I .
⟹ (x*(y*x)* z ∈ I. [Since I is a Q-Smarandache ideal of X]
⟹ (x*(y*x) ∈ I. [Snce I is a Q-Smarandache ideal of X]

Case 1: if x= y ,then x*(y*x)=x*(x*x)=x*0=x
[Since Q is a BCK- algebra ; x*x=0 and x*0=x ,∀ x ∈ Q]
⟹ x ∈ I. [Since x*(y*x) ∈ I]

Case 2 : if x≠ y, then x*(y*x) = x*y = x. [Since x*y =x]

$\Longrightarrow$ x∈ I. [Since x*(y*x) ∈ I and x*(y*x) = x]
Therefore ,I is a Q-Smarandach b - implicative ideal of X, ∀ b ∈ I . ■

Theorem (3.7) :
Let Q_1 and Q_2 be a two BCK-algebras contained in Q_2-Smarandache BH-algebra X Such that $Q_1 \subseteq Q_2$ and b ∈ X. Then every a Q_2-Smarandache b-implicative ideal of X is a Q_1-Smarandache b- implicative ideal of X **.**

Proof :
Let I be a Q_2 - Smarandache b - implicative ideal of X.
$\Longrightarrow$ I is a Q_2 - Smarandache ideal of X . [By Definition (3.1)]
$\Longrightarrow$ I is a Q_1 - Smarandache ideal of X. [By Theorem (2.14)]
Now, let x,y ∈ Q_1 and z ∈ I such that ((x*(y*x)) * z)*b ∈ I.
Since x,y ∈ Q_1 $\Longrightarrow$ x,y ∈ Q_2. [Since $Q_1 \subseteq Q_2$]
Now, we have
((x*(y*x))* z)*b) ∈ I and x,y ∈ Q_2, z ∈ I .
$\Longrightarrow$ x ∈ I . [Since I is a Q_2 – Smarandache b- implicative ideal of X]
Therefore, I is a Q_1–Smarandache b- implicative ideal of X. ■

Remark (3.8) :
The converse of Theorem (3.7) is not correct in general as in the follwing example.

Example (3.9) :
Consider the Q-Smarandache BH-algebra X={0,1,2,3,4}with binary operation "*" defind by the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

where Q_1={0,1}, Q_2 ={0,1,3} are two BCK-algebras such that $Q_1 \subseteq Q_2$. The Q-Smarandache ideal I={0,1,4} is a Q_1-Smarandache 4- implicative ideal of X, but it is not Q_2–Smarandache 4-implicative ideal of X. Since, x=3, y=0, z =1, ((3*(0*3))*1)*4=((3*0)*1)*4=(3*1)*4=1*4=1 ∈ I, but x=3 ∉ I.

Theorem (3.10):
Let I be a Q-Smarandache ideal of a Q-Smarandache BH–algebra X. Then I is a Q-Smarandache b–implicative ideal of X if and only if for all x, y ∈ X and b ∈ I , x*(y*x) ∈ I imply x ∈ I.

Proof:
Let I be a Q- Smarandache b- implicative ideal of X, $\forall$ b $\in$ I .
Now, let x*(y*x) $\in$ I.
Then x*(y*x) = (x*(y*x)) *0 =((x*(y*x)) *0)*0.
[Since Q is a BCK–algebra; x*0 = x, $\forall$x $\in$ Q]
Then, we have
((x*(y*x)) *0)) * 0 $\in$ I and 0 $\in$ I implies that x $\in$ I. [Since I is a Q-Smarandache 0- implicative ideal of X]

Conversely, suppose that I is a Q-Smarandache ideal of X and the condition is satisfied.
Let x,y $\in$ Q and z,b $\in$ I such that ((x*(y*x)* z)*b $\in$ I.
$\Rightarrow$ (x*(y*x))* z $\in$ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
$\Rightarrow$ x*(y*x) $\in$ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
$\Rightarrow$ x $\in$ I. [By hypothesis]
Therefore, I is a Q-Smarandache b- implicative ideal of X .■

Theorem (3.11) :
Let X be a positive implicative Q-Smarandache BH–algebra and I be a Q-Smarandache ideal of X such that Q * I $\subseteq$ I. Then I is a Q-Smarandache b- implicative ideal of X, $\forall$ b $\in$ I .

Proof:
Let I be a Q-Smarandache ideal of X such that Q * I $\subseteq$ I.
Now, let x, y $\in$ Q and z , b $\in$ I such that ((x*(y*x)) * z)* b$\in$ I.
$\Rightarrow$ (x*(y*x))* z $\in$ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
But (x*(y*x))* z = (x*z) *((y*x)* z). [Since X is a positive implicative BH–algebra]
Now x,y $\in$ Q $\Rightarrow$ y*x $\in$ Q, so (y*x)* z $\in$ I. [Since Q * I $\subseteq$ I]
So, we have
(x* z) *((y*x)* z) $\in$ I and ((y*x)* z) $\in$ I.
$\Rightarrow$ x* z $\in$ I. [Since I is Q-Smarandache ideal of X]
$\Rightarrow$ x $\in$ I. [Since I is Q-Smarandache ideal of X]
Therefore, I is a Q-Smarandache b- implicative ideal of X. ■

Theorem (3.12):
Let X be a bounded Q-Smarandache BH-algebra such that Q is a bounded BCK- algebra and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b- implicative ideal of X, $\forall$ b $\in$ I.
Proof :
Let I be a Q-Smarandache ideal of X.
Now, let x,y $\in$ Q and z , b $\in$ I such that ((x*(y*x)) *z)*b $\in$ I.
$\Rightarrow$ (x*(y*x))* z $\in$ I. [Since I is a Q-Smarandache ideal of X]
$\Rightarrow$ (x*(y*x) $\in$ I. [Since I is a Q-Smarandache ideal of X]
$\Rightarrow$ x $\in$ I . [Since Q is a bounded BCK algebra, by Definition (2.5)]
Therefore, I is a Q-Smarandache b- implicative ideal of X. ■

Theorem (3.13):
Let X be a Q-Smarandache BH-algebra and satisfies the following condition:

$$\forall\ x, y \in Q,\ x*y = x \quad \text{with} \quad x \neq y$$

,and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b- implicative ideal of X, $\forall$b $\in$I.

Proof:

Let I be a Q-Smarandache ideal of X.

Now, let x, y∈Q and z,b∈ I such that ((x*(y*x))*z)*b∈ I.

⇒ (x*(y*x))*z∈I. [Since I is a Q-Smarandache ideal of X]

⇒ x*(y*x)∈I. [Since I is a Q-Smarandache ideal of X]

Now, we have two cases:

Case 1: if x=y, then x*(x*x) = x*0= x.

[Since Q is a BCK-algebra; x*x=0 and x*0=x]

⇒ x∈ I . [Since x*(y*x) ∈ I]

⇒ I is a Q-Smarandache b- implicative ideal of X.

Case 2: if x ≠ y, then x*(y*x) = x*y=x. [Since x*y=x]

⇒ x ∈ I. [Since x*(y*x) ∈ I]

Therefore, I is a Q-Smarandache b- implicative ideal of X.■

Theorem (3.14):

Let X is a Q-Smarandache BH-algebra X and satisfies the condition:

∀ x, y∈Q ; x = x *(y *x)

, and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b- implicative ideal of X, ∀ b ∈I.

Proof:

Let I be a Q-Smarandache ideal of X .

Now, let x, y∈Q and z,b∈ I such that ((x*(y*x))*z)*b ∈ I.

⇒ (x*(y*x))*z∈I. [Since I is a Q-Smarandache ideal of X]

⇒ x*(y*x)∈I. [Since I is a Q-Smarandache ideal of X]

Case 1: if y=0, then x*(0*x) = x*0= x.

[Since Q is a BCK-algebra; x*0=x, 0*x=0, ∀x ∈ Q]

⇒ x ∈ I.

Hence I is a Q-Smarandache implicative ideal of X. ■

Case 2: if y ≠ 0, then x*(y*x) = x. [By condition; x = x *(y *x)]

⇒ x ∈ I. [Since x*(y*x) ∈ I]

Therefore, I is a Q-Smarandache b- implicative ideal of X.■

Proposition (3.15):

Let $\{I_i ;\ i\in \lambda\}$ be famiy of a Q-Smarandache b- implicative ideals of a Q-Smarandache BH-algebra. Then $\bigcap_{i\in\lambda} I_i$ is a Q-Smarandache b- implicative ideal of X .

Proof :

Let x,y ∈ Q and $z \in \bigcap_{i\in\lambda} I_i$ such that $(x*(y*x)) * z)*b\in \bigcap_{i\in\lambda} I_i$.

⟹ $((x*(x,y)) * z) * b \in I_i$ and $z \in I_i, \forall i\in \lambda$.

⟹ $x \in I_i, \forall i \in \lambda$.

[Since I_i is a Q -Smarandache b- implicative ideal of X, $\forall\ i \in \lambda$]

⟹ $x \in \bigcap_{i\in\lambda} I_i$.

Therefore , $\bigcap_{i\in\lambda} I_i$ is a Q –Smarandache b- implicative ideal of X .■

Remark (3.16):
The union of a Q-Smarandache implicatives ideals with repect to an element b of a Q-Smarandache BH-algebra may not be a Q-Smarandache implicative ideal of X as in the following example .

Example (3.17) :
Consider the Q-Smarandache BH-algebra X={0,1,2,3,4,5} with binary operation "*" defind by the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

where Q={0,2} is a BCK-algebra. I={0,1} and J ={0,5} are two a Q-Smarandache 0-implicative ideals of X, but $I \cup J$ ={0,1,5} is not a Q-Smarandache 0- implicative ideal of X, since x=2 ,y= 0, z =5, ((2*(0*2))*5)*0=((2*0)*5)*0 =(2*5)*0 = 1*0=1 $\in$ I, but $2 \notin I \cup J$.

Proposition (3.18) :
Let $\{I_i, i\in \lambda\}$ be chain of a Q- Smarandache b- implicative ideal of a Q-Smarandache BH-algebra X. Then $\bigcup_{i\in\lambda} I_i$ is a Q-Smarandache b- implicative ideal of X .

Proof :
Since $\{I_i, i\in \lambda\}$ is a be chian of a Q- Smarandache ideal of X. Then $\bigcup_{i\in\lambda} I_i$ is a Q-Smarandache ideal of X .

Let x,y $\in$ Q and $z \in \bigcup_{i\in\lambda} I_i$ such that ((x*(y*x)) * z)*b $\in \bigcup_{i\in\lambda} I_i$ and $z \in \bigcup_{i\in\lambda} I_i$.
There exist $I_j , I_k \in \{I_i , i\in\lambda\}$ such that ((x* (y*x) * z)*b$\in I_j$ and $z \in I_k$.
$\Rightarrow$ either $I_j \subseteq I_k$ or $I_k \subseteq I_j$. [Since $\{I_i\}_{i\in\lambda}$ is chain]
$\Rightarrow$ either ((x*(y*x))* z)*b $\in I_j$ and $z \in I_k$ or ((x*(y*x))* z)*b $\in I_k$ and $z \in I_j$.
$\Rightarrow$ either $x \in I_j$ or $x \in I_k$. [Since I_j and I_k are Q -Smarandache b- implicative ideal of X]
$\Rightarrow x \in \bigcup_{i\in\lambda} I_i$.
Therefore, $\bigcup_{i\in\lambda} I_i$ is a Q-Smarandache b- implicative ideal of X .■

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