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Differentiation of the Al-Tememe Transformation And Solving Some Linear Ordinary Differential Equation With or Without Initial Conditions

مشتقة تحويل التميمي وحل بعض المعادلات التفاضلية الخطية الاعتيادية الخاضعة وغير الخاضعة لشروط ابتدائية

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Abstract:
Our aim in this paper is to find the derivative of Al-Tememe transformation and to find Al-Tememe transformation by using derivative for new functions and to use it to solve some Linear Ordinary Differential Equation that have variable coefficients with or without initial conditions.

المستخلص:
هدفنا في هذا البحث هو ايجاد المشتقة لتحويل التميمي وايجاد تحويل التميمي لدوال جديدة بأستخدام المشتقة وايجاد حل المعادلات التفاضلية الاعتيادية الخطية ذات المعاملات المتغيرة الخاضعة وغير الخاضعة لشروط ابتدائية.

Introduction:
Laplace transform [3] is an important method for solving ODEs with constant coefficients under initial conditions.
The Al-Tememe transform [4] began at 2008 by prof A.H.Mohammed , this transform used to solve ordinary differential equation with variable coefficients with initial conditions. We will use the idea which exist at [3] to solve ODEs contains the terms $(ln\,x)^n$ and $[x^m\,(ln\,x)^n]$; where $m, n \in \mathbb{N}$ by using Al-Tememe transform.

Basic definitions and concepts :

Definition 1: [2]
Let f is defined function at a period (a, b) then the integral transformation for f whose it's symbol $F(p)$ is defined as :

$$F(p) = \int_a^b k(p, x) f(x) dx$$

Where k is a fixed function of two variables, which is called the kernel of the transformation, and a, b are real numbers or $\mp\infty$, such that the integral above convergent.

Definition 2: [4]
The Al-Tememe transformation for the function $f(x)$; $x > 1$ is defined by the following integral :

$$\mathcal{T}[f(x)] = \int_1^\infty x^{-p} f(x) dx = F(p)$$

Such that this integral is convergent , p positive is constant.

Property 1: [4]
Al-Tememe transformation is characterized by the linear property, that is:
$\mathcal{T}[Af(x)+Bg(x)] = A\mathcal{T}[f(x)] + B\mathcal{T}[g(x)]$,
Where A , B are constants ,the functions $f(x)$, $g(x)$ are defined when $x > 1$.
The Al-Tememe transform of some fundamental functions are given in table(1) [4] :

ID	Function ,$\boldsymbol{f(x)}$	$\boldsymbol{F(p) = \int_1^\infty x^{-p} f(x) dx} = \boldsymbol{\mathcal{T}[f(x)]}$	Regional of convergence
1	$k; k = constant$	$\frac{k}{p-1}$	$\boldsymbol{p > 1}$
2	x^n , $n \in R$	$\frac{1}{p-(n+1)}$	$\boldsymbol{p > n+1}$
3	$ln\, x$	$\frac{1}{(p-1)^2}$	$\boldsymbol{p > 1}$
4	$x^n\, ln\, x , n \in R$	$\frac{1}{[p-(n+1)]^2}$	$\boldsymbol{p > n+1}$
5	$sin\,(a\, ln\, x)$	$\frac{a}{(p-1)^2 + a^2}$	$\boldsymbol{p > 1}$
6	$cos\,(a\, ln\, x)$	$\frac{p-1}{(p-1)^2 + a^2}$	$\boldsymbol{p > 1}$
7	$sinh\,(a\, ln\, x)$	$\frac{a}{(p-1)^2 - a^2}$	$\boldsymbol{\lvert p-1 \rvert > a}$
8	$cosh\,(a\, ln\, x)$	$\frac{p-1}{(p-1)^2 - a^2}$	$\boldsymbol{\lvert p-1 \rvert > a}$

Table (1).
From the Al-Tememe definition and the above table, we get:
Theorem 1:
If $\mathcal{T}[\, f(x)] = F(p)$ and a is constant, then $\mathcal{T}[\, x^{-a} f(x)] = F(p+a)$ .see [4]
Definition 3: [4]
Let $f(x)$ be a function where ($x > 1$) and $\mathcal{T}[f(x)] = F(p), f(x)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}[F(p)] = f(x)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function.
Property 2: [4]
If $\mathcal{T}^{-1}[F_1(p)\,] = f_1(x)$, $\mathcal{T}^{-1}[F_2(p)\,] = f_2(x)$,..., $\mathcal{T}^{-1}[F_n(p)\,] = f_n(x)$ and $a_1, a_2, \dots, a_n$ are constants then,

$$\mathcal{T}^{-1}[a_1F_1(p) + a_2F_2(p) + \cdots + a_nF_n(p)] = a_1f_1(x) + a_2f_2(x) + \cdots + a_nf_n(x)$$

Definition 4: [5]
The equation

$$a_0x^n \frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = f(x),$$

Where $a_0, a_1, \dots, a_n$ are constants and $f(x)$ is a function of x , is called ***Euler's equation***.
Theorem 2: [4]
If the function $f(x)$ is defined for $x > 1$ and its derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$ are exist then:

$$\mathcal{T}\left[x^n f^{(n)}(x)\right] = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \cdots - (p-n)\big(p-(n-1)\big)\dots(p-2)f(1) + (p-n)!\,F(p)$$

We will use Theorem(2) to prove that

$\mathcal{T}(\ln x)^n = \dfrac{n!}{(p-1)^{n+1}}; \quad n \in \mathbb{N}$

if $\quad n = 1 \quad \Rightarrow \mathcal{T}(\ln x) = \dfrac{1}{(p-1)^2}$ (Table1) ... (1)

If $\quad n = 2 \quad \Rightarrow \quad y = (\ln x)^2 \quad \Rightarrow y(1) = 0$

$y' = 2\ln x \cdot \dfrac{1}{x} = \dfrac{2}{x}\ln x \quad \Rightarrow \quad xy' = 2\ln x$

$\mathcal{T}(xy') = 2\mathcal{T}(\ln x) = 2 \cdot \dfrac{1}{(p-1)^2} = \dfrac{2}{(p-1)^2}$

$\because \mathcal{T}(xy') = -y(1) + (p-1)\mathcal{T}(y) \quad \Rightarrow \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$

$\therefore \ (p-1)\mathcal{T}(y) = \dfrac{2}{(p-1)^2} \quad \Rightarrow \quad \mathcal{T}(y) = \dfrac{2}{(p-1)^3} = \dfrac{2!}{(p-1)^3}$ ... (2)

If $\quad n = 3 \quad \Rightarrow \quad y = (\ln x)^3 \quad \Rightarrow y(1) = 0$

$y' = 3(\ln x)^2 \cdot \dfrac{1}{x} = \dfrac{3}{x}(\ln x)^2 \quad \Rightarrow \quad xy' = 3(\ln x)^2$

$\mathcal{T}(xy') = 3\,\mathcal{T}(\ln x)^2 = 3.\dfrac{2}{(p-1)^3} = \dfrac{6}{(p-1)^3}$

$\because \ \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$

$(p-1)\mathcal{T}(y) = \dfrac{6}{(p-1)^3} \quad \Rightarrow \mathcal{T}(y) = \dfrac{6}{(p-1)^4} = \dfrac{3!}{(p-1)^4}$ ... (3)

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Also, $\quad y = (\ln x)^n \quad \Rightarrow \ y(1) = 0$

$y' = n(\ln x)^{n-1} \cdot \dfrac{1}{x} \quad \Rightarrow \quad xy' = n\ (\ln x)^{n-1}$

$\mathcal{T}(xy') = n\,\mathcal{T}(\ln x)^{n-1} = n \cdot \dfrac{(n-1)!}{(p-1)^n} = \dfrac{n!}{(p-1)^n}$

$\because \ \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$

$\therefore (p-1)\mathcal{T}(y) = \dfrac{n!}{(p-1)^n} \quad \Rightarrow \mathcal{T}(y) = \dfrac{n!}{(p-1)^{n+1}}$ ... (n)

$$\boxed{\therefore \ \mathcal{T}(\ln x)^n = \frac{n!}{(p-1)^{n+1}}; \quad n \in \mathbb{N}}$$

Also we will use Theorem(2) to find $\mathcal{T}[x^m (\ln x)^n]$; $\quad n, m \in \mathbb{N}$

The first case : If $\quad n = 1$

$$\mathcal{T}[x^m \ln x] = \frac{1!}{[p-(m+1)]^2} \ ; m \in \mathbb{N} \qquad \text{Table1}$$

The second case: If $\quad n = 2$ To find $\mathcal{T}[x^m (\ln x)^2]$

If $\quad m = 1 \quad \Rightarrow \mathcal{T}[x(\ln x)^2]$

Consider, $\quad y = x(\ln x)^2 \quad \Rightarrow \ y(1) = 0$

$y' = x \cdot 2(\ln x) \cdot \dfrac{1}{x} + (\ln x)^2 \quad \Rightarrow \quad xy' = 2x(\ln x) + x(\ln x)^2$

$\mathcal{T}(xy') = 2\mathcal{T}[x(\ln x)] + \mathcal{T}(y) = 2 \cdot \dfrac{1}{(p-2)^2} + \mathcal{T}(y)$

$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y) \ \Rightarrow (p-1)\mathcal{T}(y) = \dfrac{2}{(p-2)^2} + \mathcal{T}(y)$

$\Rightarrow (p-2)\mathcal{T}(y) = \dfrac{2}{(p-2)^2} \quad \Rightarrow \mathcal{T}[x\,(\ln x)^2] = \dfrac{2}{(p-2)^3}$ ... (4)

If $m = 2 \implies \mathcal{T}[x^2\,(\ln x)^2]$

Consider, $y = x^2\,(\ln x)^2 \implies y(1) = 0$

$$y' = x^2 \cdot 2(\ln x) \cdot \frac{1}{x} + 2x(\ln x)^2 \implies xy' = 2x^2(\ln x) + 2x^2(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^2(\ln x)] + 2\mathcal{T}(y) = 2 \cdot \frac{1}{(p-3)^2} + 2\mathcal{T}(y)$$

$$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y) \implies (p-1)\mathcal{T}(y) = \frac{2}{(p-3)^2} + 2\mathcal{T}(y)$$

$$\implies (p-3)\mathcal{T}(y) = \frac{2}{(p-3)^2} \quad \implies \mathcal{T}[x^2\,(\ln x)^2] = \frac{2}{(p-3)^3} \qquad \dots(5)$$

If $m = 3 \implies \mathcal{T}[x^3\,(\ln x)^2]$

Consider, $y = x^3\,(\ln x)^2 \implies y(1) = 0$

$$y' = x^3 \cdot 2(\ln x) \cdot \frac{1}{x} + 3x^2(\ln x)^2 \implies xy' = 2x^3(\ln x) + 3x^3(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^3(\ln x)] + 3\mathcal{T}(y) = 2 \cdot \frac{1}{(p-4)^2} + 3\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 2 \cdot \frac{1}{(p-4)^2} + 3\mathcal{T}(y) \implies \mathcal{T}[x^3\,(\ln x)^2] = \frac{2}{(p-4)^3} \qquad \dots(6)$$

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$\mathcal{T}[x^m\,(\ln x)^2] \quad ; m \in \mathbb{N} \quad \Rightarrow y = x^m\,(\ln x)^2 \implies y(1) = 0$

$$y' = x^m \cdot 2(\ln x) \cdot \frac{1}{x} + m x^{m-1}(\ln x)^2 \implies xy' = 2x^m(\ln x) + m x^m(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^m(\ln x)] + m\mathcal{T}(y) = 2 \cdot \frac{1}{[p-(m+1)]^2} + m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 2 \cdot \frac{1}{[p-(m+1)]^2} + m\mathcal{T}(y)$$

$$\Rightarrow \mathcal{T}[x^m\,(\ln x)^2] = \frac{2!}{[p-(m+1)]^3} \quad ; m \in \mathbb{N} \qquad \dots(m)$$

Note: These cases are also true for $m \in \mathbb{Q}$

The third case: To find $\mathcal{T}[x^m\,(\ln x)^3]$

If $m = 1 \Rightarrow \mathcal{T}[x(\ln x)^3]$

Consider, $y = x(\ln x)^3 \implies y(1) = 0$

$$y' = x \cdot 3(\ln x)^2 \cdot \frac{1}{x} + (\ln x)^3 \implies xy' = 3x(\ln x)^2 + x(\ln x)^3$$

$$\mathcal{T}(xy') = 3\mathcal{T}[x(\ln x)^2] + \mathcal{T}(y) = 3 \cdot \frac{2}{(p-2)^3} + \mathcal{T}(y)$$

$$\implies (p-1)\mathcal{T}(y) = \frac{3!}{(p-2)^3} + \mathcal{T}(y)$$

$$\implies (p-2)\mathcal{T}(y) = \frac{3!}{(p-2)^3} \quad \implies \mathcal{T}[x\,(\ln x)^2] = \frac{3!}{(p-2)^4} \qquad \dots(7)$$

If $m = 2 \implies \mathcal{T}[x^2\,(\ln x)^3]$

Consider, $y = x^2\,(\ln x)^3 \implies y(1) = 0$

$$y' = x^2 \cdot 3(\ln x)^2 \cdot \frac{1}{x} + 2x(\ln x)^3 \implies xy' = 3x^2(\ln x)^2 + 2x^2(\ln x)^3$$

$$\mathcal{T}(xy') = 3\mathcal{T}[x^2(\ln x)^2] + 2\mathcal{T}(y) = 3 \cdot \frac{2}{(p-3)^3} + 2\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y)=3\cdot\frac{2}{(p-3)^3}+2\mathcal{T}(y)\ \Rightarrow \mathcal{T}(x^2\,(\ln x)^3)=\frac{3!}{(p-3)^4} \qquad \dots(8)$$

If $m=3 \Rightarrow \mathcal{T}[x^3\,(\ln x)^3]$

Consider, $y=x^3\,(\ln x)^3 \quad \Rightarrow \quad y(1)=0$

$$y'=x^3\cdot 3(\ln x)^2\cdot\frac{1}{x}+3x^2(\ln x)^3$$

$$xy'=3x^3(\ln x)^2+3x^3(\ln x)^3 \Rightarrow \mathcal{T}(xy')=3\mathcal{T}[x^3(\ln x)^2]+3\mathcal{T}(y)$$

$$=3\cdot\frac{2}{(p-4)^3}+3\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y)=3\cdot\frac{2}{(p-4)^3}+3\mathcal{T}(y)\ \Rightarrow \mathcal{T}[x^3\,(\ln x)^3]=\frac{3!}{(p-4)^4} \qquad \dots(9)$$

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$\mathcal{T}[x^m\,(\ln x)^3]$

consider $\Rightarrow y=x^m\,(\ln x)^3 \quad \Rightarrow \quad y(1)=0$

$$y'=x^m\cdot 3(\ln x)^2\cdot\frac{1}{x}+mx^{m-1}(\ln x)^3 \Rightarrow xy'=3x^m(\ln x)^2+mx^m(\ln x)^3$$

$$\mathcal{T}(xy')=3\mathcal{T}[x^m(\ln x)^2]+m\mathcal{T}(y)=3\cdot\frac{2}{[p-(m+1)]^3}+m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y)=3\cdot\frac{2}{[p-(m+1)]^3}+m\mathcal{T}(y)$$

$$\Rightarrow \mathcal{T}[x^m\,(\ln x)^3]=\frac{3!}{[p-(m+1)]^4}\ ;m\in\mathbb{N} \qquad \dots(h)$$

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Gradually we find now

$$\mathcal{T}[x^m\,(\ln x)^n]\ ;m\in\mathbb{N}\ ,n\in\mathbb{N}$$

Consider, $y=x^m\,(\ln x)^n \quad \Rightarrow \quad y(1)=0$

$$y'=x^m\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}+mx^{m-1}(\ln x)^n \Rightarrow xy'=nx^m(\ln x)^{n-1}+mx^m(\ln x)^n$$

$$\mathcal{T}(xy')=n\mathcal{T}[x^m(\ln x)^{n-1}]+m\mathcal{T}(y)=n\cdot\frac{(n-1)!}{[p-(m+1)]^n}+m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y)=n\cdot\frac{(n-1)!}{[p-(m+1)]^n}+m\mathcal{T}(y)$$

$$\boxed{\mathcal{T}[x^m\,(\ln x)^n]=\frac{n!}{[p-(m+1)]^{n+1}}\ ;\ m\in\mathbb{N}\ ,n\in\mathbb{N}}$$

Definition* 5*: [3] :A function f has exponential order α if there exist constants $M>0$ and α such that for some $x_0\geq 0$

$$|f(x)|\leq M\,e^{\alpha x}, \qquad x\geq x_0.$$

Definition* 6*: [6]

A function $f(x)$ is piecewise continuous on an interval $[a,b]$ if the interval can be partitioned by a finite number of points $a=x_0<x_1<\cdots<x_n=b$ such that:

1. $f(x)$ is continuous on each subinterval (x_i,x_{i+1}), for $i=0,1,2,\dots,n-1$
2. The function f has jump discontinuity at x_i , thus

$$\left|\lim_{x\to x_i^+} f(x)\right| < \infty \,, i = 0,1,2,\dots,n-1 \,; \left|\lim_{x\to x_i^-} f(x)\right| < \infty \,, \qquad i = 0,1,2,\dots,n$$

***Note**: A function is piecewise continuous on* $[0,\infty)$ *if it is piecewise continuous in* $[0,a]$ *for all* $a > 0$

Differentiation of the Laplace Transform [1]

Let f be piecewise continuous function on $[0,\infty)$ of exponential order α and $\mathcal{L}[f(x)] = F(p)$ then :

$$\boxed{\frac{d^n}{dp^n}F(p) = \mathcal{L}[(-1)^n x^n f(x)] \;\; ; \;\; n = 1,2,3,\dots \quad (p > \alpha)}$$

Differentiation of Al-Tememe transformation

Theorem 3 : Let f be piecewise continuous function on $[1,\infty)$ of order α and $\mathcal{T}[f(x)] = F(p)$ then:

$$\boxed{\frac{d^n}{dp^n}\mathcal{T}[f(x)] = (-1)^n \mathcal{T}[(\ln x)^n f(x)] \;\; ; \;\; n = 1,2,3,\dots \quad (p > \alpha)}$$

Proof:

$$\mathcal{T}[f(x)] = \int_1^\infty x^{-p} f(x)dx$$

$$\Rightarrow \frac{d}{dp}\mathcal{T}[f(x)] = \frac{d}{dp}\int_1^\infty x^{-p} f(x)dx = \int_1^\infty \frac{\partial}{\partial p}x^{-p} f(x)dx$$

Note: $\frac{\partial}{\partial p}x^{-p}$; $x\ constant = -x^{-p}\ln x$

$$= \int_1^\infty (-x^{-p}\ln x) f(x)dx = (-1)^1 \int_1^\infty \ln x \cdot x^{-p} f(x)dx$$

Hence,

$$\frac{d}{dp}\mathcal{T}[f(x)] = (-1)^1 \mathcal{T}[\ln x \cdot f(x)] \qquad \dots(10)$$

Also,

$$\frac{d^2}{dp^2}\mathcal{T}[f(x)] = \frac{d}{dp}\left[\frac{d}{dp}\mathcal{T}[f(x)]\right] = \frac{d}{dp}\left[(-1)^1\mathcal{T}[\ln x \cdot f(x)]\right]$$

$$= (-1)^1 \frac{d}{dp}\int_1^\infty \ln x \cdot x^{-p} f(x)dx$$

$$= (-1)^1 \int_1^\infty \frac{\partial}{\partial p} \cdot \ln x \cdot x^{-p} f(x)dx$$

$$= (-1)^1 \int_1^\infty \frac{\partial}{\partial p} x^{-p} \cdot \ln x \cdot f(x)dx$$

$$= (-1)^1 \int_1^\infty (-x^{-p}\ln x) \cdot \ln x \cdot f(x)dx$$

$$= (-1)^2 \int_1^\infty (\ln x)^2 \cdot x^{-p} f(x)\, dx$$

So,

$$\frac{d^2}{dp^2}\mathcal{T}[f(x)] = (-1)^2 \mathcal{T}[(\ln x)^2 \cdot f(x)] \qquad \dots(11)$$

And so,on

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$$\frac{d^n}{dp^n}\mathcal{T}[f(x)] = (-1)^{n-1}\int_1^{\infty}\frac{\partial}{\partial p}(\ln x)^{n-1}\cdot x^{-p}\cdot f(x)dx$$
$$= (-1)^{n-1}\int_1^{\infty}(\ln x)^{n-1}\,(-1\cdot x^{-p}\cdot \ln x)\,f(x)dx$$
$$= (-1)^{n}\int_1^{\infty}(\ln x)^{n}\cdot x^{-p}\cdot f(x)\,dx$$
$$= (-1)^{n}\mathcal{T}[(\ln x)^{n}\cdot f(x)] \qquad \dots(n)$$

$$\boxed{\frac{d^n}{dp^n}\mathcal{T}[f(x)] = (-1)^{n}\mathcal{T}[(\ln x)^{n}f(x)]\ ;\ n = 1,2,3,\dots \quad (p > \alpha)\blacksquare.}$$

$$\Rightarrow \mathcal{T}[(\ln x)^{n}f(x)] = (-1)^{n}\frac{d^n}{dp^n}\mathcal{T}[f(x)]\ ;\ n = 1,2,3, (p > \alpha)$$

***Example* 1:** To find $\mathcal{T}[x^{-3}\ln x]$ we note that

$$\mathcal{T}[\ln x\cdot x^{-3}] = (-1)^{1}\frac{d}{dp}\mathcal{T}(x^{-3})$$
$$= -\frac{d}{dp}\cdot\left(\frac{1}{p+2}\right) = \frac{1}{(p+2)^2}$$

By using the previous conversion table

$$\mathcal{T}[\ln x\cdot x^{-3}] = \mathcal{T}[x^{-3}\cdot \ln x] = \frac{1}{(p+2)^2}$$

***Example* 2:** To find $\mathcal{T}[\ln x.\sin(3\ln x)]$ by applying the above relationship:

$$\mathcal{T}[\ln x.\sin 3\ln x] = (-1)^{1}\frac{d}{dp}\left[\frac{3}{(p-1)^2+9}\right]$$
$$= \frac{-3\cdot\{-2(p-1)\}}{[(p-1)^2+9]^2} = \frac{6(p-1)}{[(p-1)^2+9]^2}$$

***Example* 3:** To find $\mathcal{T}[\ln x\,.\cosh\ (5\ln x)]$ by applying the above relationship:

$$\mathcal{T}[\ln x\ \cdot\cosh\ (5\ln x)] = \frac{-d}{dp}\left[\frac{(p-1)}{(p-1)^2-25}\right]$$
$$= -\frac{[(p-1)^2-25]-(p-1)\cdot 2(p-1)}{[(p-1)^2-25]^2}$$
$$= \frac{p^2-2p+26}{[(p-1)^2-25]^2} = \frac{(p-1)^2+25}{[(p-1)^2-25]^2}$$

***Differential Equations with Polynomial Coefficients by using Laplace Transform* .**[3]**.**

$\mathcal{L}[xf(x)] = -F'(p),$
$\mathcal{L}[xf'(x)] = -pF'(p) - F(p),$
$\mathcal{L}[xf''(x)] = -p^2F'(p) - 2pF(p) + y(0)$

Differential Equations with multiple of polynomial and logarithms coefficients by using Al-Tememe Transform: Recall (Theorem) that for

$$F(p) = \mathcal{T}[f(x)]$$
$$\frac{d^n}{dp^n}\mathcal{T}[f(x)] = (-1)^{n}\mathcal{T}[(\ln x)^{n}f(x)]\ ;\ n = 1,2,3,\dots \quad (p > \alpha)$$

For a piecewise continuous function $f(x)$ on $[1,\infty)$ and of exponential order α.Hence, for $n = 1$

$$\mathcal{T}[ln\,x\,.\,y'] = \frac{-d}{dp}\mathcal{T}(y')$$

$$\mathcal{T}[ln\,x\,.\,xy'] = \frac{-d}{dp}\mathcal{T}(xy')$$

$$= \frac{-d}{dp}[-y(1) + (p-1)\mathcal{T}(y)]$$

$$= -[(p-1)\mathcal{T}'(y) + \mathcal{T}(y)]$$

$$\boxed{\mathcal{T}[ln\,x\,.\,xy'] = -(p-1)\mathcal{T}'(y) - \mathcal{T}(y)}$$

Similarly, for

$$\mathcal{T}[ln\,x\,.\,x^2y''] = -\frac{d}{dp}[-y'(1) - (p-2)y(1) + (p-1)(p-2)\mathcal{T}(y)]$$

$$= -[-y(1) + (p-2)(p-1)\mathcal{T}'(y) + (2p-3)\mathcal{T}(y)]$$

$$\boxed{\mathcal{T}[ln\,x\,.\,x^2y''] = y(1) - (p-2)(p-1)\mathcal{T}'(y) - (2p-3)\mathcal{T}(y).}$$

In many cases these formulas for $\mathcal{T}[ln\,x\,.\,xy']$ and $\mathcal{T}[ln\,x\,.\,x^2y'']$ are useful to solve linear differential equations with variable coefficients .

Example 4: To solve the differential equation:

$$xlnx.\,y' - y = (lnx)^2$$

By taking ($\mathcal{T}$) to both sides we get:

$\mathcal{T}[xln\,x\,.\,y'] - \mathcal{T}(y) = \mathcal{T}[(lnx)^2]$

$-(p-1)\mathcal{T}'(y) - \mathcal{T}(y) - \mathcal{T}(y) = \dfrac{2!}{(p-1)^3}$

$-(p-1)\mathcal{T}'(y) - 2\mathcal{T}(y) = \dfrac{2}{(p-1)^3}$

$\Longrightarrow\ \mathcal{T}'(y) + \dfrac{2}{p-1}\mathcal{T}(y) = \dfrac{-2}{(p-1)^4}$

This is an ordinary linear differential equation and its integrating factor is given by:

$\mu = e^{\int\frac{2}{p-1}dp} = e^{2\ln(p-1)} = e^{ln(p-1)^2} = (p-1)^2$

Therefore,

$[\mathcal{T}(y).\,(p-1)^2]' = \dfrac{-2}{(p-1)^2}$

And

$\mathcal{T}(y).\,(p-1)^2 = -2\displaystyle\int \frac{1}{(p-1)^2}\,dp$

$\mathcal{T}(y).\,(p-1)^2 = -2.\dfrac{-1}{(p-1)} + c$

$\mathcal{T}(y) = \dfrac{2}{(p-1)^3} + \dfrac{c}{(p-1)^2}$

By taking $\mathcal{T}^{-1}$ to both sides we get :

$$y = \mathcal{T}^{-1}\left[\frac{2}{(p-1)^3}\right] + \mathcal{T}^{-1}\left[\frac{c}{(p-1)^2}\right]$$

$$\boxed{y = (ln\,x)^2 + cln\,x \quad ; \; c \text{ constant}}$$

***Example* 5:** To solve the differential equation:

$$\ln x . x^2 y'' + xy' = x\ln x \quad ; \; y(1) = 1$$

By taking ($\mathcal{T}$) to both sides we get:

$\mathcal{T}(\ln x . x^2 y'') + \mathcal{T}(xy') = \mathcal{T}(x\ln x)$

$y(1) - (p-2)(p-1)\mathcal{T}'(y) - (2p-3)\mathcal{T}(y) - y(1) + (p-1)\mathcal{T}(y) = \dfrac{1}{(p-2)^2}$

$-(p-2)(p-1)\mathcal{T}'(y) - (2p-3-p+1)\mathcal{T}(y) = \dfrac{1}{(p-2)^2}$

$-(p-2)(p-1)\mathcal{T}'(y) - (p-2)\mathcal{T}(y) = \dfrac{1}{(p-2)^2}$

$\mathcal{T}'(y) + \dfrac{1}{p-1}\mathcal{T}(y) = \dfrac{-1}{(p-2)^3(p-1)}$

This is an ordinary linear differential equation and its integrating factor is given by:

$\mu = e^{\int \frac{1}{p-1} dp} = e^{\ln(p-1)} = (p-1)$

Therefore,

$[\mathcal{T}(y).(p-1)]' = \dfrac{-1}{(p-2)^3}$

$\Longrightarrow \; \mathcal{T}(y).(p-1) = \displaystyle\int \frac{-1}{(p-2)^3} dp$

$\qquad = \dfrac{1}{2(p-2)^2} + C$

$\mathcal{T}(y) = \dfrac{1}{2(p-2)^2(p-1)} + \dfrac{C}{(p-1)}$

$\dfrac{1}{2(p-2)^2(p-1)} = \dfrac{A}{p-2} + \dfrac{B}{(p-2)^2} + \dfrac{C}{p-1}$

$\qquad = \dfrac{-1/2}{p-2} + \dfrac{1/2}{(p-2)^2} + \dfrac{1/2}{p-1}$

$\therefore \mathcal{T}(y) = \dfrac{-1/2}{p-2} + \dfrac{1/2}{(p-2)^2} + \dfrac{1/2}{p-1} + \dfrac{C}{p-1}$

By taking $\mathcal{T}^{-1}$ to both sides we get :

$y = -1/2 \; \mathcal{T}^{-1}\left(\dfrac{1}{p-2}\right) + 1/2 \; \mathcal{T}^{-1}\left[\dfrac{1}{(p-2)^2}\right] + 1/2 \; \mathcal{T}^{-1}\left(\dfrac{1}{p-1}\right) + \; \mathcal{T}^{-1}\left(\dfrac{C}{p-1}\right)$

And,

$y = -1/2\, x + 1/2\, x\ln x + 1/2 + c$

$y = -1/2\, x + 1/2\, x\ln x + c_1 \;\; ; \;\; c_1 = 1/2 + c$

Since, $\qquad y(1) = 1 \;\; \Longrightarrow \;\; c_1 = 3/2$

$$\boxed{y = -1/2\, x + 1/2\, x\ln x + 3/2}$$

Conclusion: apply Al-Tememe transformations to solve linear ordinary differential equations with variable coefficients by using initial and without any initial conditions.

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