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D_{rb}-Sets And Associated Separation Axioms

مجموعات D_{rb} وبديهيات الفصل المرافقة

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Abstract

The purpose of the present paper is to introduce and investigate new classes of Separation Axioms called Associated Separation Axioms by using the concepts of rb-open and D_{rb}-sets. Several characterizations and fundamental properties concerning D_{rb}-sets and Associated Separation Axioms are obtained. Furthermore, the relationships between this Associated Separation Axioms and other types of Separation Axioms are also discussed.

الملخّص

إنّ غرض البحث الحالي أن يقدّم ويدرس أصناف جديدة من بديهيات الفصل سميت بديهيات الفصل المرافقة وذلك باستعمال مفاهيم المجموعات المفتوحة من نوع - rb ومجموعات D_{rb} . قدمت وأثبتت عدّة خواص ومميزات أساسية تتعلق بالمجموعات المفتوحة من نوع – rb ومجموعات D_{rb} وبديهيات الفصل المرافقة . علاوة على ذلك ،درست ونوقشت العلاقات بين هذه الأنواع من بديهيات الفصل .

Introduction

The notion of Regular closed sets are introduced and studied recently by Stone [1].Levine, [2] initiated the study of generalized closed sets in topological spaces in1970. Palaniappan and Rao [3]and Sharmistha Bhattacharya [4] , introduced and investigated regular generalized Closed sets. b-closed sets have been introduced and investigated by Andrijevi'c [5],[6] and [7] . The notion of regular b-Closed (briefly rb-Closed) sets is introduced and studied recently by Nagaveni and Narmadha [8],[9] and [10]. Al- Omari and Noorani [11] investigated the Class of generalized b-Closed sets and obtained some of its fundamental properties. In 1982, Tong [12] introduced the notion of D-sets and used these sets to introduce a separation axiom D_1 which is strictly between T_0 and T_1 . By using the concept rb-open We introduce and study the concepts of a new type of Associated Separation Axioms is called D_{rb}-Sets Associated Separation Axioms .

1.Preliminaries

Throughout this paper, a space X means a topological space on which no separation axioms are assumed unless otherwise mentioned, for a subset A of a space (X,τ), Cl(A) and Int(A) denote the Closure of A and interior of A, respectively. A^C denotes the complement of A in X . A subset A of a space (X, τ) is called a regular open set if A= int (Cl(A)) and a regular Closed set if A= Cl(int (A)), The intersection of all regular closed sets of X containing A is called the regular closure of A and is denoted by rCl(A). b-open [5] if $A \subset Cl(int(A)) \cup int(Cl(A))$. The complement of a b-open set is said to be b-Closed. The intersection of all b-Closed sets of X containing A is called the b-Closure of A and is denoted by bCl(A). The union of all b-open sets of X contained in A is called b-interior of A and is denoted by bInt(A). A subset A of a space X is said to be generalized Closed (briefly g-Closed) if Cl (A) $\subset$ U whenever A$\subset$ U and U is open in X [2], A is called regular-generalized closed (briefly rg-closed) if Cl (A) $\subset$ U whenever A $\subset$ U and U is regular-open in X [3], A is called generalized b-Closed (briefly gb-Closed) if bCl (A) $\subset$ U whenever A $\subset$ U and U is open in X [11], A is called regular b-Closed(briefly rb-Closed) if rCl (A) $\subset$ U whenever A $\subset$U and U is b-open in X [9] . The complement of g-closed (rg-closed, gb-closed,rb-closed)is called g-open (rg-open, gb-open, rb-open). The family of all rb-open (rb-Closed) subsets of a space X is denoted by RBO(X), RBC(X)) and the collection of all rb-open(rb-closed) subsets of X containing a fixed point x is denoted by RBO(X,x)(RBC(X,x)), the end of the proof is denoted by◄ .

Definition 1.1[12]. A subset A of a topological space (X, τ) is called D-set if there are two open sets U and V such that $U \neq X$ and $A=U-V$.

Definition 1.2. A point $x \in X$ which has only X as the rb-neighborhood is called a rb- neat point.

Definition 1.3. A topological space (X, τ) is rb-symmetric if for x and y in X , $x \in Cl_{rb}(\{y\})$ implies $y \in Cl_{rb}(\{x\})$.

Lemma 1.4.[10]. For subsets A and A_α ($\alpha \in \Lambda$) of a space (X ,τ) , the following are hold:
(i) If $A \subset B$, then $Cl_{rb}(A) \subset Cl_{rb}(B)$.
(ii) $Cl_{rb} (\cap\{A_\alpha : \alpha \in \Lambda\}) \subset \cap\{Cl_{rb} (A_\alpha) : \alpha \in \Lambda\}$.
(iii) $Cl_{rb} (\cup\{A_\alpha : \alpha \in \Lambda\}) =\cup \{Cl_{rb} (A_\alpha) : \alpha \in \Lambda\}$.

Definition 1.5. Let A be a subset of topological space (X,τ), The rb−kernel of A, denoted by $Ker_{rb}(A)$ is defined to be the set $Ker_{rb} (A) = \cap\{H \in RBO (X ,\tau): A \subset H\}$.

Lemma 1.6. Let (X,τ) be a topological space and $x \in X$, then $Ker_{rb} (A) = \{x \in X : Cl_{rb} (\{x\}) \cap A \neq \phi\}$.
Proof: Let $x \in Ker_{rb} (A)$ and suppose $Cl_{rb} (\{x\}) \cap A = \phi$. Hence $x \notin [Cl_{rb} (\{x\})]^C$ which is a rb−open set containing A. This is contradiction, since $x \in Ker_{rb} (A)$.
Consequently, $Cl_{rb} (\{x\}) \cap A \neq \phi$. Next, let $Cl_{rb} (\{x\}) \cap A \neq \phi$ and suppose that $x \notin Ker_{rb} (A)$;Then there exists an rb-open set H containing A and $x \notin H$. Let $y \in Cl_{rb} (\{x\}) \cap A$. Hence, H is a rb-neighborhood of y which $x \notin H$. This contradiction with $x \in Ker_{rb} (A)$ ◄

Lemma 1.7. Let (X ,τ) be a topological space and $x \in X$, then $y \in Ker_{rb} (\{x\})$ if and only if $x \in Cl_{rb} (\{y\})$.
Proof: Suppose that $y \notin Ker_{rb} (\{x\})$, then there exists an rb -open set G containing x such that $y \notin G$; therefore, we have $x \notin Cl_{rb} (\{y\})$. The proof of converse case can be done similarly◄

Lemma 1.8. The following statements are equivalent for any points x and y in a topological space (X ,τ) :
(i) $Ker_{rb} (\{x\}) \neq Ker_{rb} (\{y\})$.
(ii) $Cl_{rb} (\{x\}) \neq Cl_{rb} (\{y\})$.
Proof: (i) $\rightarrow$ (ii) Let $Ker_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$,then there exist $x_0 \in X$ such that $x_0 \in Ker_{rb}(\{x\})$ and $x_0 \notin Ker_{rb}(\{y\})$, since $x_0 \in Ker_{rb}(\{x\})$ it follows that $\{x\}\cap CL_{rb}\{x_0\} \neq \varphi$, from this we get $x \in CL_{rb}(\{x_0\})$, by $x_0 \notin Ker_{rb}(\{y\})$, we have $\{y\} \cap CL_{rb}(\{x_0\}) = \varphi$, since $x \in CL_{rb}(\{x_0\})$, $CL_{rb}(\{x\}) \subset CL_{rb}(\{x_0\})$ and $\{y\} \cap CL_{rb}(\{x\}) = \varphi$; therefore $CL_{rb}(\{x\}) \neq CL_{rb}(\{y\})$.
(ii) $\rightarrow$ (i) Suppose $CL_{rb}(\{x\}) \neq CL_{rb}(\{y)\}$, then there exist $y \in X$ such that $y\in CL_{rb}(\{x\})$ and $y \neq CL_{rb}(\{y\})$, then ther exist rb−open set containing x_0 and therefore x but not y , namely, $y \notin Ker_{rb}(\{x\})$,hence $Ker_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$◄

Definition1.9. A topological function $f : (X,\tau) \rightarrow (Y,\tau)$ is called rb−Closed if the image of every rb−Closed subset of X is rb−Closed in Y.

Definition 1.10. A net χ_n in a topological space (X,τ) is called rb-converges to x if χ_n is eventually in every rb-open set containing x .

Lemma 1.11. Let x and y be any two points in X such that every net in X rb−converging to y rb−converges to x, then $x \in Cl_{rb}(\{y\})$.

Proof: suppose $x_\alpha = y$ for each $\alpha \in \Lambda$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ is a net in $Cl_{rb}(\{y\})$, since $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to y, then $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to x and this implies that $x \in Cl_{rb}(\{y\})$◀

2. D_{rb}-sets and associated separation axioms

Definition 2.1. A subset H of a topological space (X, τ) is called a D_{rb}- set if there are two sets G_1 , $G_2 \in RBO(X,\tau)$ such that $G_1 \neq X$ and $H=G_1-G_2$.

Example 2.2. Let X = {a, b, c}. τ = {X, ϕ, {a}, {b}, {a, b}, {b, c}}and A = {a,b}, then A is D_{rb}- set.

Definition 2.3. A topological space (X, τ) is said to be

(i) rb-D_0 if for any distinct pair of points x and y of X there exist a D_{rb}-set of X containing x but not y or a D_{rb}-set of X containing y but not x,

(ii) rb-D_1 if for any distinct pair of points x and y of X there exist a D_{rb}-set of X containing x but not y and a D_{rb}-set of X containing y but not .

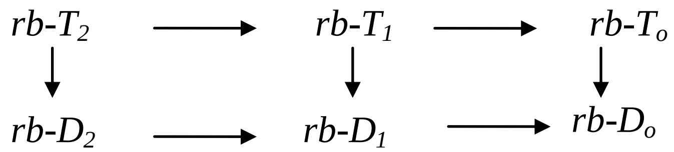
(iii) rb-D_2 if for any distinct pair of points x and y of X there exists disjoint D_{rb}-sets G and H of X containing x and y, respectively.

(iv) rb-T_0 if for any distinct pair of points x and y in X, there exist rb-open set U in X containing x but not y or rb-open set V in X containing y but not x.

(v) rb-T_1 if for any distinct pair of points x and y in X, there exist rb-open set U in X containing x but not y and rb-open set V in X containing y but not x .

(vi) rb-T_2 if for any distinct pair of points x and y in X, there exist rb-open sets U and V in X containing x and y, respectively, such that $U \cap V=\emptyset$.

Remark 2.4. the following diagram explain relationship between the above definitions in (2.3)



Example 2.5. Let X = {a, b, c}. τ = {X, ϕ, {a}, {b}, {a, b}} is rb-D_0 but not rb-D_1 .

Theorem 2.6. In a topological space the following statements are true:

(i) (X,τ) is rb − T_0 if and only if it is rb − D_0.

(ii) (X,τ) is rb – D_2 if and only if it is rb – D_1.

Proof: (i) The sufficiency is stated in Remark (2.3).

Necessarily, let X be rb − D_0, then for each distinct pair of points $x, y \in X$, at least one of x, y, say x belongs to D_{rb}-set H but $y \notin H$, let $H= H_1 - H_2$ where $H_1 \neq X$ and $H_1, H_2 \in RBO(X, \tau)$, then $x \in H_1$ and for $y \notin H$ we have two cases : first $x \in H_1$ but $y \notin H_1$, second $y \in H_2$ but $x \notin H_2$. Hence X is rb − T_0 .

(ii) sufficiency, Clearly in Remark (2.2) .

Necessarily , suppose X is rb − D_1, then for each distinct pair $x, y \in X$, we have D_{rb}-sets U,V such that $x \in U$,$y \notin U$ and $y \in V$, $x \notin V$. Let $U =H_1 - H_2$ and $V = G_1 - G_2$ and H_1 , H_2 ,G_1 ,$G_2 \in RBO(X, \tau)$. From $x \notin V$, it follows that either $x \notin G_1$ or $x \in G_1$ and $x \in G_2$. We have two cases separately :first $x \notin G_1$. By $y \notin U$ we have two subcases:

(a) $y \notin H_1$. From $x \in H_1 - H_2$, it follows that $x \in H_1 - (H_2 \cup G_1)$ and by $y \in G_1 - G_2$ we have $y \in G_1 - (H_1 \cup G_2)$. Therefore $(H_1 - (H_2 \cup G_1)) \cap (G_1 - (H_1 \cup G_2) = \varphi$.

(b) $y \in H_1$ and $y \in H_2$. We have $x \in H_1 - H_2$, $y \in H_2$,$(H_1 - H_2) \cap H_2 =\varphi$.

(2) $x \in G_1$ and $x \in G_2$. We have $y \in G_1 - G_2$, $x \in G_2$,$(G_1 - G_2) \cap G_2 = \varphi$;therefore, X is rb − D_2◄

Theorem 2.7. A topological space (X, τ) is rb-T_0 if and only if for each pair of distinct points x, y of X, $CL_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$.

Proof: Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$,
let $z \in X$ such that $z \in Cl_{rb}(\{x\})$ but $z \notin Cl_{rb}(\{y\})$. We Claim that $x \notin Cl_{rb}(\{y\})$, if $x \in Cl_{rb}(\{y\})$ then $Cl_{rb}(\{x\}) \subset Cl_{rb}(\{y\})$, this contradicts with
$z \notin Cl_{rb}(\{y\})$, consequently x belongs to the rb-open set $[Cl_{rb}(\{y\})]^C$ to which y does not belong.
Necessity. Let (X, τ) be an rb-T_0 space and x, y be any two distinct points of X, there exists an rb-open set G containing x or y, then G^C is an rb-Closed set which $x \notin G^C$ and $y \in G^C$, since $Cl_{rb}(\{y\})$ is the smallest rb-Closed set containing y (Corollary 1.1), $Cl_{rb}(\{y\}) \subset G^C$, and $x \notin Cl_{rb}(\{y\})$, hence $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$◄

Theorem 2.8. Let (X, τ) be an rb−T_0 topological space ; then (X, τ) is rb-D_1 if and only if has no rb-neat point.

Proof: Sufficiency , since X is rb-D_1, so each point x of X is contained in a D_{rb}-set $H=G_1-G_2$ and thus in G_1. By definition $G_1 \neq X$. This implies that x is not a rb− neat point.
Necessity ,if X is rb − T_0 ;then for each distinct pair of points x and $y \in X$, at least one of them, x has an rb-neighborhood H containing x and not y . Thus H which is different from X is an D_{rb}-set . If X has no rb-neat point ; then y is not a rb-neat point . This means that there exists an rb-neighborhood G of y such that $G \neq X$. Thus $y \in G - H$ but not x and G –H is an D_{rb}-set. Hence X is rb-D_1◄

Theorem 2.9. For a topological space (X, τ), the following properties are equivalent:
(i) (X, τ) is rb-symmetric .
(ii) For each $x \in X$, {x} is rb-Closed .
(iii) (X, τ) is rb-T_1 .

Proof : (i)↔(ii) let $x \in CL_{rb}(\{y\})$ but $y \notin CL_{rb}(\{x\})$. This means that $[CL_{rb}(\{x\})]^C$ contains y, this implies that $CL_{rb}(\{y\})$ is a subset of $[CL_{rb}(\{x\})]^C$, now $[CL_{rb}(\{x\})]^C$ contains x this is a contradiction.
Suppose that $\{x\} \subset G \in RBO(X)$ but $CL_{rb}\{x\}$ is not a subset of G, therefore $CL_{rb}\{x\}$ and G^C are not disjoint, let y belongs to their intersection, now we have $x \in CL_{rb}\{y\}$ which is a subset of G^C and $x \notin G$, but this is a contradiction.
(ii)→(iii) suppose {x} is rb-Closed for every $x \in X$, let $x, y \in X$ with $x \neq y$ .Now $x \neq y$ implies $y \in \{x\}^C$, hence$\{x\}^C$ is an rb-open set containing y but not x, similarly $\{y\}^C$ is an rb-open set containing x but not y. Accordingly X is an rb-T_1 space.
(iii)→(i) Suppose that $y \notin CL_{rb}(\{x\})$, then, since $x \neq y$, by (iii) there exists an rb-open set G containing x such that $y \notin G$ and hence $x \notin CL_{rb}(\{y\})$. This shows that $x \in CL_{rb}(\{y\})$ implies $y \in CL_{rb}(\{x\})$;therefore, (X, τ) is rb-symmetric◄

Theorem 2.10. The following four properties are equivalent:
(i) X is rb-T_2 .
(ii) Let $x \in X$. For each $y \neq x$, there exists an rb−open set H such that $x \in H$ and $y \notin CL_{rb}(H)$.
(iii) For each $x \in X$, $\cap\{CL_{rb}(H) : H \in RBO(X, \tau) \text{ and } x \in H\} = \{x\}$.
(iv) The diagonal $\Delta = \{(x, x) : x \in X\}$ is rb−Closed in X × X}.

Proof: (i) → (ii) Let $x \in X$ and $y \neq x$; Then there are disjoint rb−open sets H and G such that $x \in H$ and $y \in G$. Clearly, G^C is rb−Closed, $Cl_{rb}(H) \subset G^C$, $y \notin G^C$ and therefore $y \notin Cl_{rb}(H)$.
(ii) → (iii) If $y \neq x$, then there exists an rb−open set H such that $x \in H$ and $y \notin Cl_{rb}(H)$. So $y \notin \cap \{Cl_{rb}(H) : U \in RBO(X, \tau) \text{ and } x \in H\}$.

(iii) $\to$ (iv) We prove that Δ^C is rb−open , let $(x, y) \notin \Delta$; then $y \neq x$ and since $\cap\{CL_{rb}(H) : H \in RBO(X, \tau) \text{ and } x \in H\} = \{x\}$,there is some $H \in RBO(X, \tau)$ with $x \in H$ and $y \notin Cl_{rb}(H)$. Since $H \cap (Cl_{rb}(H))^C = \phi$, $H \times (Cl_{rb}(H))^C$ is an rb−open set such that $(x, y) \in H \times (CL_{rb}(H))^C \subset \Delta^C$.
(iv) $\to$(i) If $y \neq x$; then $(x, y) \notin \Delta$ and thus there exist rb−open sets H and G such that $(x, y) \in H \times G$ and $(H \times G) \cap \Delta = \phi$. Clearly, for the rb−open sets H and G we have : $x \in H$, $y \in G$ and $H \cap G = \phi$ ◂

3. Applications

Definition 3.1. A topological function $f: (X, \tau_1) \to (Y, \tau_2)$ is called
(i) rb− continuous if $f^{-1}(H)$ is rb− Closed in (X, τ_1) for every Closed set H of (Y,τ_2) .
(ii) rb− continuous if $f^{-1}(G)$ is rb− open in (X, τ_1) for every open set G of (Y,τ_2) .
(iii) rb− irresolute if $f^{-1}(H)$ is rb- Closed in(X, τ_1) for every rb−Closed set H in (Y, τ_2).
(iv) rb− irresolute if $f^{-1}(G)$ is rb−open in(X, τ_1) for every rb−open set G in (Y, τ_2).

Theorem 3.2. If topological function $f : (X, \tau)\to(Y, \tau_2)$ is a rb-continuous surjective function and H is a D-set of (Y, τ_2); then the inverse image of H is a D_{rb}-set of (X, τ_1).

Proof: Let G_1 and G_2 be two open sets of (Y, τ_2) and $H=G_1 - G_2$ be a D-set and $G_1 \neq Y$. We have $f^{-1}(G_1)$ and $f^{-1}(G_1) \in RBO(X, \tau_1)$ such that $f^{-1}(G_1) \neq X$. Hence $f^{-1}(H)=f^{-1}(G_1 - G_2)= f^{-1}(G_1)- f(G2)$, this is implies to $f^{-1}(H)$ is an D_{rb}-set◂

Theorem 3.3. If topological function $f : (X, \tau_1) \to (Y, \tau_2)$ is a rb-irresolute surjection and H is a D_{rb}-set in Y; then the inverse image of H is an D_{rb}-set in X.

Proof: Let H be a D_{rb}-set in Y; then there exist rb-open sets G_1 and G_2 in Y such that $H= G_1- G_2$ and $G_1 \neq Y$. Since f is rb-irresolute; then $f^{-1}(G_1)$ and $f^{-1}(G_1) \in RBO(X, \tau_1)$ and since $G_1 \neq Y$, we have $f^{-1}(G_1) \neq X$. Hence $f^{-1}(H) = f^{-1}(G_1- G_2) = f^{-1}(G_1)- f^{-1}(G_2)$ is an D_{rb}-set◂

Theorem 3.4. If $f : (X, \tau_1) \to (Y, \tau_2)$ is an rb-continuous injective function and (Y, τ_2) is a D_1 −space ; then (X, τ_1) is a rb−D_1−space.

Proof: Let Y is a D_1−space and x , y be any pair of distinct points in X, since f is injective and Y is a D_1−space; then there exists D-sets G_x and G_y of Y containing f(x) and f(y) respectively such that $f(x)\notin G_y$ and $f(y)\notin G_x$. By theorem (4.1) $f^{-1}(G_x)$ and $f^{-}(G_y)$ are D_{rb}-sets in X containing x and y respectively such that $x\notin f^{-1}(G_y)$ and $y\notin f^{-1}(G_x)$; therefore, X is a rb−D_1−space◂

Theorem 3.5. If $f : (X,\tau_1)\to (Y,\tau_2)$ is rb-irresolute and injective and (Y,τ_2) is rb−D_1 space; then (X,τ_1) is rb−D_1.

Proof: Assume (Y,τ_2) is rb-D_1 and f is injective rb−irresolute, let x , y be any pair of distinct points of X. Since f is injective and Y is rb-D_1 space, then there exists D_{rb}-sets G_x and G_y of Y containing f(x) and f(y) respectively such that $f(y)\notin G_x$ and $f(x)\notin G_y$. By theorem 4.2 **,** $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are D_{rb}−sets in X containing x and y respectively. Hence X is rb-D_1 space◂

Theorem 3.6. A topological space (X, τ_1) is a rb−D_1 if for each pair of distinct points $x , y \in X$; then there exists a rb-continuous surjective topological function $f: (X,\tau_1) \to (Y,\tau_2)$ where (Y,τ_2) is a D_1− space such that f(x) and f(y) are distinct.

Proof: Let x and y be any pair of distinct points in X, from hypothesis, there exists an rb-continuous surjective function f of a space (X, τ_1) onto a D_1- space (Y,τ_2) such that $f(x) \neq f(y)$. Hence there exists disjoint D-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y)\in G_y$. Since f is rb-continuous and surjective, by theorem 4.1 then $f^{-1}(G_x)$ and $f^{-1}(G_y)$are disjoint D_{rb}−sets in X containing x and y respectively. Hence (X, τ_1) is an rb-D_1-space◂

Theorem 3.7. A topological space **(**X,τ_1) is rb-D_1 if and only if for each pair of distinct points $x , y\in X$, there exists an rb-irresolute surjective topological function $f: (X,\tau_1) \to (Y,\tau_1)$,where (Y,τ_1) is rb- D_1 space such that f(x) and f(y) are distinct.

Proof: Necessity , for every pair of distinct points $x , y \in X$, it suffices to take the identity function on X.

Sufficiency, let $x \neq y \in X$. By hypothesis ,there exists an rb-irresolute, surjective function from X onto an rb-D_1 space such that $f(x) \neq f(y)$. Hence there exists disjoint $D_{rb}-$ sets G_x, $G_y \subset Y$ such that $f(x) \in G_x$ and $f(y) \in G_y$, since f is rb-irresolute and surjective, by theorem 4.2 , $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint D_{rb}-sets in X containing x and y respectively; therefore, X is rb-D_1 space◄

Theorem 3.8. If a topological function $f : (X, \tau_1) \to (Y, \tau_2)$ is injective, contra rb-continuous and Y is a Urysohn space, then X is rb-T_2.

Proof: Let $x,y \in X$ with $x \neq y$,since f is injective , then $f(x) \neq f(y)$,since Y is a Urysohn space, there exists open sets G_1 and G_2 in Y such that $f(x) \in G_1$, $f(y) \in G_2$ and $Cl(G_1) \cap CL(G_2) = \emptyset$, since f is contra rb-continuous, by Theorem 4.2 there exists rb-open sets A and B in X such that $x \in A$, $y \in B$ and $f(A) \subset CL(G_1)$, $f(B) \subset CL(G_2)$, then $f(A) \cap f(B) = \emptyset$ and so $f(A \cap B)=\emptyset$,this implies that $A \cap B = \emptyset$ and hence X is rb-T_2◄

Theorem 3.9. Let f be a topological continuous function from a space (X,τ_1) to an rb-T_1-space (Y ,τ_2) , then f has an rb-strongly Closed graph.

Proof: Let $(x,y) \in X \times Y$ with $(x,y) \notin G(f)$. This means that $f(x) \neq y$,and since Y is an rb-T_1-space, there exists rb-open sets $V \subset Y$ such that $y \in CL(V)$ and $f(x) \notin CL(V)$, that is, $f(x) \in Y - CL(V)$, by continuity of f at x, there exists U open set in a space X contain x such that $f(U) \subset Y - CL(V)$, this implies that $f(U) \cap CL(V)=\emptyset$ and it follows that $(x,y) \in U \times CL(V)$ with $(U \times CL(V)) \cap G(f)=\emptyset$, Thus G(f) is rb-strongly Closed◄

Definition 3.10. Let (X, τ) be a topological space, we define the rbr-h$(X, \tau) = \{f \mid f : (X, \tau) \to (X, \tau)$ is a rb-irresolute bijection, $f^{-1} : (X, \tau) \to (X, \tau)$ is rb-irresolute$\}$.

Theorem 3.11. Let (X, τ) be a topological space, the collection rbr-h(X, τ) forms a group under the composition of functions.

Proof : If $f : (X, \tau) \to (X, \tau_1)$ and $g : (X, \tau_1) \to (X, \tau_2)$ are rbr homeomorphisms, then clearly the composition $g \circ f : (X, \tau) \to (X, \tau_2)$ is a rbr-homeomorphism, it is obvious that for a bijective rbr-homeomorphism $f : (X, \tau) \to (X, \tau 1)$, $f^{-1} : (X, \tau 1) \to (X, \tau)$ is also an rbr-homeomorphism and the identity $I : (X, \tau) \to (X, \tau)$ is a rbr-homeomorphism. A binary operation α : rbr-h$(X, \tau) \times$ rbr-h$(X, \tau) \to$ rbr-h(X, τ) is well defined by $\alpha(a, b) = b \circ a$, where a, b $\in$ rbr-h(X, τ) and $b \circ a$ is the composition of a and b. By using the above properties, the set rbr-h(X, τ) is a group under composition of functions◄

4.rb− R_0 spaces and rb − R_1 spaces

Definition 4.1. A topological space (X ,τ) is said to be:

(i) Sober rb -R_0 if $\cap_{x\in X} Cl_{rb}(\{x\}) = \phi$.

(ii) rb -R_0 space if every rb-open set contains the rb-Closure of each of its singletons .

(iii) rb -R_1 if for x , y in X with $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$, there exist disjoint rb-open sets G and H such that $Cl_{rb}(\{x\})$ is a subset of G and $Cl_{rb}(\{y\})$ is a subset of H .

Theorem 4.2. A topological space (X ,τ) is sober rb -R_0 if and only if $Ker_{rb}(\{x\}) \neq X$ for every $x \in X$.

Proof: Suppose that the space X be sober rb -R_0 and that there is a point y in X such that $Ker_{rb}(\{y\}) = X$, then $y \notin H$ which H is some proper rb-open subset of X , this implies that $y \in \cap_{x\in X} Cl_{rb}(\{x\})$, but this is a contradiction .

Consequently , $Ker_{rb}(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap_{x\in X} Cl_{rb}(\{x\})$, then every rb-open set containing y must contain every point of X , this implies that the space X is the unique rb-open set containing y , hence $Ker_{rb}(\{y\}) = X$ which is a contradiction , therefore, (X,τ) is sober rb -R_0◄

Theorem 4.3. If the topological space X is sober rb -R_0 and Y is any topological space , then the product $X \times Y$ is sober rb -R_0 .

Proof: We must showing that $\cap_{(x, y)\in X\times Y} Cl_{rb}(\{x, y\}) = \phi$, we have $\cap_{(x, y)\in X\times Y} Cl_{rb}(\{x, y\}) \subset \cap_{(x, y)\in X\times Y} (Cl_{rb}(\{x\}) \times Cl_{rb}(\{y\}))= \cap_{x\in X} Cl_{rb}(\{x\}) \times \cap_{y\in Y} Cl_{rb}(\{y\}) \subset \phi \times Y = \phi$◄

Theorem 4.4. If a topological function $f : (X,\tau) \rightarrow (Y,\tau)$ is an bijective and rb-Closed function and X is sober rb$-R_0$, then Y is sober rb$-R_0$.

Proof: Is Clearly from definitions (4.1) and (1.9)◄

Theorem 4.5. If a topological space (X,τ) is rb – R_1 ,then (X,τ) is rb – R_0 topological space.

Proof: Let $x \in G$ and $G \in RBO(X,\tau)$, if $y \notin G$,then since $x \notin Cl_{rb}(\{y\})$, $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$, hence there exist $H_y \in RBO(X,\tau)$ such that $Cl_{rb}(\{y\} \subset H_y$ and $x \notin H_y$, which implies $y \notin Cl_{rb}(\{x\})$, thus $Cl_{rb}(\{x\}) \subset G$, therefore, X is rb $- R_0$ ◄

Theorem 4.6. A topological space (X,τ) is rb$-R_1$ if and only if for x , y in (X,τ) with $Ker_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$,there exist disjoint rb$-$open sets G and H such that $Cl_{rb}(\{x\}) \subset G$ and $Cl_{rb}(\{y\}) \subset H$.

Proof: It follows from above lemma and definition (4.1)◄

Theorem 4.7. A topological space (X,τ) is rb $- R_0$ if and only if for any element $x, y \in (X,\tau)$ and $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$, then $Cl_{rb}(\{x\}) \cap Cl_{rb}(\{y\}) = \varphi$.

Proof: Necessity: let (X,τ) is rb $- R_0$ and $x, y \in X$ such that $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$, then there exist $x_0 \in Cl_{rb}(\{x\})$ and $x_0 \notin Cl_{rb}(\{y\})$ or $x_0 \in Cl_{rb}(\{y\})$ and $x_0 \notin Cl_{rb}(\{x\})$, there exists $G \in RBO(X, \tau)$ such that $y \notin G$ and $x_0 \in G$,then $x \in G$,therefore, we have $x \notin Cl_{rb}(\{y\})$, thus $x \in [Cl_{rb}(\{y\})]^C \in RBO(X, \tau)$, which implies $Cl_{rb}(\{x\}) \subset [Cl_{rb}(\{y\})]^C$ and $Cl_{rb}(\{x\}) \cap Cl_{rb}(\{y\}) = \varphi$. the proof for otherwise is similar.

Sufficiency : let $x \in G$ such that $G \in RBO(X, \tau)$ and $y \notin G$, this mean $y \in G^C$, then $x \neq y$ and $x \notin Cl_{rb}(\{y\})$,this shows that $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$, from assumption, $Cl_{rb}(\{x\}) \cap Cl_{rb}(\{y\}) = \varphi$, hence $y \notin Cl_{rb}(\{x\})$,therefore, $Cl_{rb}(\{x\}) \subset G$ ◄

Theorem 4.8. A topological space (X,τ) is rb$-R_0$ if and only if for any two points $x, y \in X$ and $KerCl_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$, then $Ker_{rb}(\{x\}) \cap Ker_{rb}(\{y\})= \varphi$.

Proof: let (X,τ) be rb $- R_0$,from (Lemma 4.3) for any two points $x, y \in X$ if $Ker_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$, then $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$,by theorem(4.6) we get $Ker_{rb}(\{x\}) \cap Ker_{rb}(\{y\}) = \varphi$, let $x_0 \in Ker_{rb}(\{x\}) \cap Ker_{rb}(\{y\})$, then $x_0 \in Ker_{rb}(\{x\})$ and lemma 4.2, implies that $x \in Cl_{rb}(\{x_0\})$, since $x \in Cl_{rb}(\{x\})$,then by Theorem 4.6 $Cl_{rb}(\{x\}) = Cl_{rb}(\{x_0\})$. Similarly, we have $Cl_{rb}(\{y\}) = Cl_{rb}(\{x_0\}) = Cl_{rb}(\{x\})$, this is a contradiction, therefore, we have $Ker_{rb}(\{x\}) \cap Ker_{rb}(\{y\}) = \varphi$.

Conversely :assume $Cl_{rb}(\{x\}) \neq Cl_{rb}(\{y\})$ for each $x, y \in X$, then by (Lemma 4.3) $Ker_{rb}(\{x\}) \neq Ker_{rb}(\{y\})$,since $Ker_{rb}(\{x\}) \cap Ker_{rb}(\{y\}) = \varphi$, then $Cl_{rb}(\{x\}) \cap Cl_{rb}(\{y\}) = \varphi$, by $x_0 \in Cl_{rb}(\{x\})$, thus implies $x \in Ker_{rb}(\{x_0\})$ and therefore $Ker_{rb}(\{x\}) \cap Ker_{rb}(\{x_0\}) \neq \varphi$,(by Theorem 4.6) (X,τ) is a rb$- R_0$ space◄

Theorem 4.9. For a topological space (X,τ) The following properties are equivalent:

(i) (X,τ) is a rb $- R_0$ space.

(ii) For any $A \neq \varphi$ and $G \in RBO(X, \tau)$ such that $A \cap G \neq \varphi$, there exist $H \in RBC(X, \tau)$ such that $A \cap H \neq \varphi$ and $H \subset G$.

(iii) Any $G \in RBO(X, \tau)$,$G = \cup\{H \in RBC(X, \tau) : H \subset G\}$.
(iv) Any $H \in RBC(X, \tau)$, $H = \cap\{G \in RBO(X, \tau) : H \subset G\}$.
(v) For any $x \in (X,\tau)$,then $Cl_{rb}(\{x\}) \subset Ker_{rb}(\{x\})$.

Proof: (i) →(ii) Let A be a nonempty set of X and $G \in RBO(X, \tau)$ such that $A\cap G \neq \varphi$, there exists $x \in A \cap G$, since $x\in G$ and $G \in RBO(X, \tau)$, $Cl_{rb}(\{x\}) \subset G$, the set $H = Cl_{rb}(\{x\})$, then $H\in RBC(X, \tau)$, $H \subset G$ and $A\cap H\neq\varphi$

(ii)→ (iii) Let $G \in RBO(X, \boldsymbol{\tau})$, then $\cup\{H \in RBC(X, \tau) :H \subset G\} \subset G$, let x_0 be any point of G, there exists $H \in RBC(X, \boldsymbol{\tau})$ such that $x_0 \in H$ and $H \subset G$, therefore, we have $x_0 \in H \subset \cup\{H \in RBC(X, \tau) :H\subset G\}$ and hence $G = \cup\{H \in RBC(X, \tau) :H\subset G\}$.

(iii) →(iv) This is obvious.

(iv) → (v) Let x be any point of X and $y \notin Ker_{rb}(\{x\})$,there exists $G_1\in RBO(X, \boldsymbol{\tau})$ such that $x \in G_1$ and $y \notin G_1$, then $Cl_{rb}(\{y\}) \cap G_1 = \varphi$, by (iv) $(\cap\{G\in RBO(X, \tau)\text{: } Cl_{rb}(\{y\}) \subset G\}) \cap G_1 = \varphi$, there exist $G\in RBO(X,\boldsymbol{\tau})$ such that $x \notin G$ and $Cl_{rb}(\{y\}) \subset G$, hence $Cl_{rb}(\{x\}) \cap G = \varphi$ and $y \notin Cl_{rb}(\{x\})$, Consequently, we obtain $Cl_{rb}(\{x\})\subset Ker_{rb}(\{x\})$.

(v) →(i) Let $x \in G$ and $G\in RBO(X, \tau)$, assume $y \in Ker_{rb}(\{x\})$, then $x\in Cl_{rb}(\{y\})$ and $y\in G$, this implies that $Cl_{rb}(\{x\}) \subset Ker_{rb}(\{x\}) \subset G$, hence $(X,\tau)$is $rb - R_0$◀

Corollary 4.10. For a topological space(X,τ) The following properties are equivalent:
(i) (X,τ) is a $rb – R_0$ space .
(ii) $Cl_{rb}(\{x\}) = Ker_{rb}(\{x\})$ for all $x \in (X,\tau)$.

Proof: (i)→(ii) Suppose X is $rb-R_0$, by Theorem 4.8, $Cl(\{x\}) = Ker_{rb}(\{x\})$ for all $x \in X$,let $y \in Ker_{rb}(\{x\})$, then $x \in Cl_{rb}(\{y\})$ and by Theorem 4.6 we get $Cl_{rb}(\{x\}) = Cl_{rb}(\{y\})$,therefore, $y \in Cl_{rb}(\{x\})$ and hence $Ker_{rb}(\{x\}) \subset Cl_{rb}(\{x\})$, this shows that $Cl_{rb}(\{x\}) = Ker_{rb}(\{x\})$.

(ii)→(i): This is obvious by Theorem 4.8◀

Theorem 4.11. For a topological (X,τ) the following properties are equivalent:
(i) (X,τ) is a $rb - R_0$ space.
(ii) $x \in Cl_{rb}(\{y\})$ if and only if $y \in Cl_{rb}(\{x\})$, for any points x and y in X.

Proof: (i) →(ii) Let $x \in Cl_{rb}(\{y\})$ and $y \in G\in RBO(X,\tau)$, now by hypothesis, $x \in G$, therefore, every rb−open set which contain y contains x, hence $y \in Cl_{rb}(\{x\})$.

(ii)→(i) Let $x \in G$ and $G \in RBO(X,\tau)$, if $y\notin G$, then $x\notin Cl_{rb}(\{y\})$ and hence $y\notin CL_{rb}(\{x\})$, this implies that $Cl_{rb}(\{x\}) \subset G$, hence $(X,\tau)$is $rb-R_0$, we observed that by Definition 3.7 and Theorem 4.10 the notions of rb−symmetric and $rb - R_0$ are equivalent◀

Theorem 4.12. For a topological (X,τ) the following properties are equivalent:
(i) $(X,\tau)$is a $rb - R_0$ space .
(ii) If S is rb−Closed; then $S = Ker_{rb}(S)$.
(iii) If S is rb−Closed and $x \in S$, then $Ker_{rb}(\{x\}) \subseteq S$
(iv) If $x \in (X,\tau)$, then $Ker_{rb}(\{x\}) \subset Cl_{rb}(\{x\})$.

Proof: (i) → (ii) If $x \notin S$ and $S\in RBC(X)$, thus S^C is rb−open and contains x ,since X is $rb - R_0$, then $Cl_{rb}(\{x\}) \subset S^C$, this mean $Cl_{rb}(\{x\}) \cap S = \varphi$ and by lemma (2.1)(vii) $x \notin Ker_{rb}(S)$, therefore, $Ker_{rb}(S) = S$.

(ii) → (iii) If $A \subset B$ implies that $Ker_{rb}(A) \subset Ker_{rb}(B)$, therefore, it follows from (ii) that $Ker_{rb}(\{x\}) \subset Ker_{rb}(S) = S$.

(iii) → (iv) Since $x \in Cl_{rb}(\{x\})$ and $Cl(\{x\})$ is rb−Closed, by (iii) $Ker_{rb}(\{x\}) \subset Cl_{rb}(\{x\})$.

(iv)→ (i) We show the implication by using Theorem 4.10, let $x \in Cl_{rb}(\{y\})$, then by Lemma 4.2 $y \in Ker_{rb}(\{x\})$, since $x \in Cl_{rb}(\{x\})$ and $Cl_{rb}(\{x\})$ is rb−Closed, by (iv) we obtain $y \in KerCl_{rb}(\{x\}) \subset$

$Cl_{rb}(\{x\})$, therefore, $x \in Cl_{rb}(\{y\})$ implies that $y \in Cl_{rb}(\{x\})$, the converse is obvious and X is rb – R_0 ◀

Theorem 4.13. For a topological (X,τ) the following statements are equivalent:

(i) (X,τ) is a rb − R_0 space .

(ii) If x, y ∈ (X,τ), then $y \in Cl_{rb}(\{x\})$ if and only if every net in X rb−converging to y rb−converges to x.

Proof: (i) → (ii) Let x, y ∈ X such that $y \in Cl_{rb}(\{x\})$, suppose $\{x_\alpha\}_{\alpha\in\Lambda}$ is a net in (X,τ) such that $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to y, since $y \in Cl_{rb}(\{x\})$, by theorem 4.6 we have $Cl_{rb}(\{x\}) = Cl_{rb}(\{y\})$, therefore, $x \in Cl_{rb}(\{y\})$, this means that $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to x, Conversely, let x, y ∈ X such that every net in X rb-converging to y rb−converges to x, then $x \in Cl(\{y\})$, by theorem (4.6), we have $Cl_{rb}(\{x\}) = Cl_{rb}(\{y\})$, therefore, $y \in Cl_{rb}(\{x\})$.

(ii)→(i) Assume x and y are any two points of X such that $Cl(\{x\}) \cap Cl_{rb}(\{y\}) \neq \varphi$, let $x_0 \in Cl_{rb}(\{x\}) \cap Cl_{rb}(\{y\})$, so there exist a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $Cl_{rb}(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to x_0, since $x_0 \in Cl_{rb}(\{y\})$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ rb−converges to y, it follows that $y \in Cl_{rb}(\{x\})$. Similarly we obtain $x \in Cl_{rb}(\{y\})$, therefore, $Cl_{rb}(\{x\}) = Cl_{rb}(\{y\})$ and by theorem (4.6), (X,τ) is rb − R_0 ◀

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