Some Types of Normalities of Fuzzy KS-Semigroups
بعض أنواع الناظمية في أشباه الزمر الضبابية KS

Sajda Kadhum Mohammed
Faculty of Education For Girls , Kufa university, Iraq
Sajidak.mohammed@uokufa.edu.iq
Sundus Najah Jaber
Faculty of Education For Girls , Kufa university, Iraq

البحث مستل

Abstract:
In this paper, we introduce new types of normalities of fuzzy KS-Semigroups, namely normal fuzzy subks- semigroups , fuzzy normal subks- semigroup of KS- semigroup and fuzzy normal subks- semigroup of fuzzy subKS- semigroup .Also,we study some of their properties .

الخلاصة :
في هذا البحث قدمنا أنواع جديدة من الا ناظم ية في أشباه الزمر الضبابية KS سميت شبه الزمرة الجزئية KS الضبابية الناظمية و شبه الزمرة الجزئية الناظمية الضبابية من شبه الزمرة KS وشبه الزمرة الجزئية الناظمية الضبابية من شبه الزمرة الجزئية الضبابية كما درسنا وبرهنا بعض من خواصها

1.1Introduction:

The notation of BCK algebra introduced by Y.Imai and K.Ise'ki [1] in 1966 . In the same year, K. Ise'ki [2]introduced the notation of BCI algebra which is a generalization of BCK algebra. In 2006 ,Kyung Ho Kim [3] introduced a new class of algebraic structure called KS semigroup .In 2009 Jocelyns S. Paradero Vilea and Mila Cawi [4] characterized ideals of KS- Semigroups and prove some properties .In 2007 , D.R. Prince Wiliams and Husain Shamshad[5] fuzzify KS semigroup and called it fuzzy KS Semigroups and introduced the notations of fuzzy subKS-Semigroups, fuzzy KS ideal ,fuzzy KS P ideal and investigated some of their related properties. In this paper, we define three new types of normalities of fuzzy KS-Semigroups and study their properties and prove some interested propositions .

2.Preliminary

This section contains some basic concepts that we needed it in this paper.

Definition (2.1)[5]

An algebraic system $(X, *, 0)$ is called a **BCK algebra** if it satisfies the following conditions:

1) $((x * y) * (x * z)) * (z * y) = 0$,
2) $(x * (x * y)) * y = 0$,
3) $x * x = 0$,
4) $0 * x = 0$
5) if $x * y = 0$ and $y * x = 0$ then $x = y$, $\forall x, y, z \in X$.

Remarks (2.2) [6]

Let X be a BCK algebra :

- A partial ordering " $\le$ " on X can be defined by $x \le y$ if and only if $x * y = 0$.
- A BCK-algebra X has the following properties:

1) $x * 0 = x$.
2) if $x*y=0$ and $y*z=0$ imply $x*z=0$.
3) if $x*y=0$ implies $(x*z)*(y*z)=0$ and $(z*y)*(z*x)=0$.
4) $(x*y)*z=(x*z)*y$.

5) (x*y)*x=0.
6) x*(x*(x*y))=x*y.
7) if (x*y)*z=0 implies (x*z)*y=0.
8) [(x*z)*(y*z)]*(x*y)=0.
9) [((x*z)*z)*(y*z)]*[(x*y)*z]=0.

Definition(2.3)[7]

A non empty subset I of a BCK –algebra X is called an **ideal** of X if the following conditions hold :

1) $0 \in I$,
2) if $x * y \in I$ and $y \in I$ imply $x \in I$, $\forall \quad x, y \in X$.

Definition(2.4) [8]

A non empty subset I of a BCK –algebra X is called a **p- ideal** of X if the following condition hold :

1) $0 \in I$,
2) if $(x * y) * z \in I \quad and \quad y * z \in I \quad \Rightarrow x * z \in I$, $\forall \ x, y, z \in X$

Definition (2.5)[4]

A **Semigroup** is an ordered pair $(X, \cdot)$,where X is a non empty set and " ." is an associative binary operation on X .

Definition (2.6)[8]

Let X be a semigroup and x an element of X . An element e of X is a **left identity** of x if $e \cdot x = x$
, a **right identity** of x if $x \cdot e = x$, an **identity** of x if $x = e \cdot x = x \cdot e$.

Definition (2.7)[9]

A non-empty subset T of a semigroup X is a **sub semigroup** of X if it is closed under the operation of X , i.e $\forall \ a, b \in T$ then $a \cdot b \in T$.

Definition (2.8)[10]

A **semigroup homomorphism** is any mapping $f : X \rightarrow T$ where X and T are semigroups which satisfies $f(xy) = f(x) f(y)$ for all $x, y \in X$.

Definition (2.9)[5]

A **KS-semigroup** is a non-empty set X with two binary operation " ∗ " and " ." , and a constant 0 satisfies the following axioms:

1. (X, ∗, 0) is a BCK-algebra.
2. (X, .) is a semigroup,
3. x.(y ∗ z) = (x.y) ∗ (x.z) and (x ∗ y).z = (x.z) ∗ (y.z), for all x, y, z $\in X$.

Remarks (2.10)

- Throughout this paper X denotes the KS- semigroups unless otherwise specified . We shall write the multiplication $x \cdot y$ *by* xy .
- $x0 = 0x = 0 \quad \forall x \in X$ where X is a KS- semigroup,[5] .

Example (2.11)[5]

Let X = {0, 1, 2 ,3 ,4 } be a set with binary operations "*" and "." defined by the following tables:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	0	0

.	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	2
3	0	0	0	1	2
4	0	1	2	3	4

X is a KS- semigroup .

Definition (2.12)[5]

A non empty subset S of X with binary operation * and . is called **sub KS-semigroup** of X if it satisfies the following condition :

1) $x * y \in S \quad \forall \; x, y \in S$.

2) $xy \in S \quad \forall \quad x, y \in S$

Definition (2.13) [3]

A strong KS-semigroup is a KS-semigroup X satisfying : $x * y = x * xy \quad \forall \; x, y \in \mathrm{X}$

Lemma(2.14) [3]

Let X be a strong KS-semigroup then :

1) $xy * y = 0 \quad \forall \; x, y \in \mathrm{X}$.

2) $x * y = 0 \leftrightarrow x * xy = 0$, $\quad \forall \; x, y \in \mathrm{X}$.

Definition (2.15) [11] Let X be a non-empty set a fuzzy subset μ of X is a function $\mu : \mathrm{X} \rightarrow [0, 1]$.

Definition (2.16) [3]

Let X and Y be KS-semigroups a mapping $f : \mathrm{X} \rightarrow \mathrm{Y}$ is called a **KS-semigroup homomorphism** (briefly homomorphism) if $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x) f(y)$ for all $x, y \in \mathrm{X}$.

Let $f : \mathrm{X} \rightarrow \mathrm{Y}$ KS-semigroup homomorphism . then the set $\{x \in \mathrm{X} : f(x) = 0\}$ is called the **kernel of f** , and denote by **ker f** . Moreover, the set $\{f(x) \in \mathrm{Y} : x \in \mathrm{X}\}$ is called the **image of f** and denote by **Im f** .

Definition (2.17) [11]

Let μ and ν be a fuzzy sets on X . Define the fuzzy set $\mu \cap \nu$ as follows: $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in \mathrm{X}$.

Definition (2.18) [11]

Let μ and ν be a fuzzy sets on X . Define the fuzzy set $\mu \cup \nu$ as follows:

$(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in \mathrm{X}$.

Definition (2.19) [5]

Let X be a non-empty set and let μ be the fuzzy subset of X for a fixed $0 \leq t \leq 1$,Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called an **upper level set of μ** .

Proposition (2.20)[12] Let A and B are fuzzy subKS-semigroup of X. Then $A \cap B$ is a fuzzy subKS-semigroup

Proposition(2.21)

Let μ and ν be fuzzy subKS-semigroups of X then $\mu \cup \nu$ is a fuzzy subKS-semigroup of X if $\mu \subseteq \nu$ *or* $\nu \subseteq \mu$.

Proof:

Let μ and ν are the fuzzy subKS-semigroup , and let $x, y \in \mu \cup \nu$ then

$$(\mu \cup \nu)(xy) = \max\{\mu(xy), \nu((xy)\}$$
$$\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\} \qquad [by\ hypothesis]$$
$$= \min\{\max\{\mu(x), \nu(x)\}, \max\{\mu(y), \nu(y)\}\} \qquad [\mu \subseteq \nu \ or\ \nu \subseteq \mu]$$
$$= \min\{(\mu \cup \nu)(x), (\mu \cup \nu)(y)\}.$$

so ,

$$(\mu \cup \nu)(x*y) = \max\{\mu(x*y), \nu((x*y)\}$$
$$\geq \max\{\min\{\mu(x),\mu(y)\},\min\{\nu(x),\nu(y)\}\} \quad [by\ hypothesis]$$
$$= \min\{\max\{\mu(x),\nu(x)\},\max\{\mu(y),\nu(y)\}\} \quad [\mu \subseteq \nu\ or\ \nu \subseteq \mu]$$
$$= \min\{(\mu \cup \nu)(x),(\mu \cup \nu)(y)\}.$$

Hence $\mu \cup \nu$ is a fuzzy subKS-semigroup.

Definition (2. 22) [13]

Let $f : X \to Y$ be a mapping of KS-Semigroup and μ be a fuzzy subset of Y . The map μ^f is the **pre-image of** μ **under f** if $\mu^f = \mu(f(x))\ \forall x \in X$

Definition (2.23) [13]

Let λ and μ be the fuzzy subsets in a set X the **cartesian product** $\lambda\times\mu : X\times X \to [0, 1]$ is defined by $(\lambda\times\mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ for all x, y $\in X$.

Proposition(2.24)[12]

Let X be a KS-semigroup and let μ , ν be a fuzzy subKS-semigroup then μ × ν is a fuzzy subKS-semigroup .

Definition (2.25) [13]

Let ν be a fuzzy subset in X the strong fuzzy relation on X that is a fuzzy relation on is ρ_V given by $\rho_V(x, y) = \min\{\nu(x),\nu(y)\}$

Proposition (2.26)[12]

Let μ be a fuzzy subKS-semigroup then $G_\mu = \{x \in X : \mu(x) = \mu(0)\}$ is a subKS-semigroup

Proposition (2.27)[12]

Let X be a KS-semigroup, ν be fuzzy set Then ρ_V is fuzzy subKS-semigroup if and only if ν is fuzzy subKS-semigroup.

Proposition (2.28)[12]

Let X be a KS-semigroup and μ , λ be two fuzzy sets in X such that $\mu\times\lambda$ is a fuzzy subKS-semigroup of $X\times X$. Then :

1) either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$.
2) If $\mu(x) \leq \mu(0)$ *for all* $x \in X$ then either $\mu(x) \leq \lambda(0)$ *or* $\lambda(x) \leq \lambda(0)$.
3) If $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$ then either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \mu(0)$.
4) either μ or λ is a fuzzy subKS-semigroup of X .

3 Normal Fuzzy SubKS- Semigroups

In this section , we give the definition of normal fuzzy subKS-semigroup and give some properties ,like union , intersection , Cartesian product , image , G_μ and other properties of normal fuzzy subKS-semigroup, also we define maximal element of the normal fuzzy subKS-semigroup with some properties .

Definition (3.1)

A fuzzy subKS-semigroup μ of X is said to be ***normal fuzzy subKS-semigroup*** if there exists $x \in X$ such that $\mu(x) = 1$.

Remark (3.2)

A fuzzy subKS-semigroup μ of X is said to be normal fuzzy subKS-semigroup if and only if $\mu(0) = 1$.

Proof:

Let μ be a normal fuzzy subKS-semigroup of X. Then
there exists $x \in X$ *such that* $\mu(x) = 1$
since by [12] we have $\mu(0) \geq \mu(x)\ \ \forall x \in X$

so $\mu(0) \geq 1$ then $\mu(0) = 1$.
conversely , it is clear .

Example (3.3)

Let X = {0, 1, 2 } be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Define a fuzzy subset $\mu : X \to [0,1]$ by $\mu(0) = 1$ *and* $\mu(x) = 0.3 \ \forall\, (x \neq 0) \in X$ .Then μ is a normal fuzzy subKS- semigroup of X .

Proposition (3.4)

Let μ and ν be normal fuzzy subKS-semigroups of X.Then $\mu \cap \nu$ is a normal fuzzy subKS-semigroup of X .

Proof:

let μ and ν be normal fuzzy subKS-semigroups of X.Then by [proposition 2.20] $\mu \cap \nu$ is a fuzzy subKS-semigroup of X also $\mu(0) = 1$ *and* $\nu(0) = 1$ so $(\mu \cap \nu)(0) = \min\{\mu(0), \nu(0)\} = 1$,therefore $\mu \cap \nu$ is a normal fuzzy subKS-semigroup .

Proposition (3.5)

Let μ and ν be normal fuzzy subKS-semigroups of X .Then $\mu \cup \nu$ be a normal fuzzy subKS-semigroup of X if $\mu \subseteq \nu$ *or* $\nu \subseteq \mu$.

Proof: It is clear

Proposition (3.6)

Let μ and ν be a normal fuzzy subKS-semigroup .Then $\mu \times \nu$ is a normal fuzzy subKS-semigroup .

Proof:

Let μ and ν be normal fuzzy subKS-semigroup of X then ,since μ and ν are fuzzy subKS-semigroup so by [proposition 2.24] $\mu \times \nu$ is a fuzzy subKS-semigroup

Now , $(\mu \times \nu)(0,0) = \min\{\mu(0), \nu(0)\} = \min\{1,1\} = 1$ [since μ , ν are normal fuzzy subKS-semigroup].Hence $\mu \times \nu$ is normal fuzzy subKS-semigroup .

Proposition (3.7)

Let $f : X \to Y$ be a homomorphism if μ is a normal fuzzy subKS-semigroup of Y. Then μ^f is a normal fuzzy subKS-semigroup of X .

Proof:

Let μ is a normal fuzzy subKS-semigroup of Y and $x, y \in X$ .Then

$$\mu^f(xy) = \mu(f(xy)) = \mu(f(x) \cdot f(y)) \ [\text{since } f \text{ is a homomorphism}]$$
$$\geq \min\{\mu(f(x)), \mu(f(y))\} \quad [\text{since } \mu \textit{ is a fuzzy subKS} - \textit{semigroup}] \text{ .So,}$$
$$= \min\{\mu^f(x), \mu^f(x)\}$$

$\mu^f(x*y) = \mu(f(x*y)) = \mu(f(x)*f(y))$ [since f *is a* homomorphism]

$\geq \min\{\mu(f(x)), \mu(f(y))\}$ [since μ *is a fuzzy subKS − semigroup*]

$= \min\{\mu^f(x), \mu^f(x)\}$.

Thus μ^f is a fuzzy subKS-semigroup . Now , $\mu^f(0) = \mu(f(0)) = \mu(0) = 1$ [since μ is a normal fuzzy subKS-semigroup]. Hence μ^f is a normal fuzzy subKS-semigroup .

Proposition (3.8)

Let $f : X \to Y$ be epimorphism if μ^f is a normal fuzzy subKS-semigroup of X then μ is a normal fuzzy subKS-semigroup of Y .

Proof:

Let μ^f be a normal fuzzy sub KS-semigroup of X .

let $x, y \in Y$ $\exists\, a, b \in X$ $\ni$ $f(a) = x$, $f(b) = y$

$\mu(xy) = \mu(f(a)f(b)) = \mu(f(ab)) = \mu^f(ab) \geq \min\{\mu^f(a), \mu^f(b)\} = \min\{\mu(f(a)), \mu(f(b))\}$

$= \min\{\mu(x), \mu(y)\}$

Also, in similar way we have $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$. Therefore, μ is a fuzzy subKS-semigroup .

Now , $\mu(0) = \mu(f(0)) = \mu^f(0) = 1$ [sine μ^f is a normal fuzzy subKS-semigroup] .

Hence μ is a normal fuzzy subKS-semigroup .

Proposition (3.9)

Let ν be a fuzzy subset of KS-semigroup X and ρ_ν be a strong fuzzy relation on X .Then ν is a normal fuzzy subKS-semigroup if and only if ρ_ν is a normal fuzzy subKS-semigroup **.**

Proof:

Let ν be a normal fuzzy subKS-semigroup of X , So by [proposition 2.27] ρ_ν is a fuzzy subKS-semigroup $\rho_\nu(0,0) = \min\{\nu(0), \nu(0)\} = \min\{1,1\} = 1$ [since ν is a normal fuzzy] .

Therefore ρ_ν is a normal fuzzy subKS-semigroup .Conversely, Let ρ_ν is a normal fuzzy subKS-semigroup of $X \times X$ then So by [proposition 2.27] ν is a fuzzy subKS-semigroup $\sin ce$ $\rho_\nu(0,0) = 1 = \min\{\nu(0), \nu(0)\} = \nu(0)$ *then* $\nu(0) = 1$ .Hence ν is a normal fuzzy subKS-semigroup .

Proposition (3.10)

Let μ be a fuzzy subKS-semigroup of X .Then a fuzzy set μ^+ defined by

$\mu^+(x) = \mu(x) + 1 - \mu(0)$ is a normal fuzzy subKS-semigroup .

Proof:

Let μ be a fuzzy subKS-semigroup of X , to prove μ^+ is a fuzzy subKS-semigroup let $x_1, x_2 \in X$ then

$\mu^+(x_1 x_2) = \mu(x_1 x_2) + 1 - \mu(0) \geq \min\{\mu(x_1), \mu(x_2)\} + 1 - \mu(0)$

$= \min\{\mu(x_1) + 1 - \mu(0), \mu(x_2) + 1 - \mu(0)\} = \min\{\mu^+(x_1), \mu^+(x_2)\}$

and in similar way we have $\mu^{+}(x_1 * x_2) = \min\{\ \mu^{+}(x_1), \mu^{+}(x_2)\ \}$hence μ^{+} is fuzzy subKS-semigroup .Moreover $\mu^{+}(0) = \mu(0) + 1 - \mu(0) = 1 \quad \forall\ x \in X$ therefore μ^{+} is a normal fuzzy subKS-semigroup .

Easily we can prove the corollaries 3.11 -3.17

Corollary (3.11)

Let μ and μ^{+} be as in the above Proposition . Then $\mu^{+}(x_0) = 0 \quad (\textit{for some } x_0 \in X) \quad \textit{implies} \quad \mu(x_0) = 0$ .Moreover μ is a normal if and only if μ^{+} is a normal .

Corollary (3.12)

For every fuzzy subKS-semigroup μ defined on X, we have $(\mu^{+})^{+} = \mu^{+}$.Moreover if μ is a normal then $(\mu^{+})^{+} = \mu$.

Corollary (3.13)

If μ and ν are two fuzzy subKS-semigroup such that $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$ then $G_{\mu} \subseteq G_{\nu}$.

Corollary (3.14)

If μ and ν are normal fuzzy subKS-semigroup of X such that $\mu \subseteq \nu$ then $G_{\mu} \subseteq G_{\nu}$.

Remark(3.15)

1) Denoted NK(X) for the set of all normal fuzzy subKS-semigroup of X .
2) It is clear that (NK(X), $\subseteq$) , a poset .

Definition (3.16)

A fuzzy subset μ defined on KS-semigroup X is called maximal if it is non-constant and μ^{+} is a maximal element of the poset (NK(X), $\subseteq$) .

Proposition (3.17)

Let μ be a non constant normal fuzzy subKS-semigroup of X if μ is a maximal element of $(NK(X), \subseteq)$ .Then μ take only values 0 and 1 .

Proposition (3.18)

Let I be a non-empty subset of X then I is a subKS-semigroup if and only if χ_I is a normal fuzzy subKS-semigroup where

$$\chi_I = \begin{cases} 1 & if \quad x \in I \\ 0 & if \quad x \notin I \end{cases}$$

Proof:

Let I be a subKS-semigroup of X and let $x, y \in I$ such that

$x * y \in I \quad and \quad xy \in I \qquad [I \ is\ a\ subKS-semigroup\ ]$ .Then

$\chi_I\ (x * y) = 1 \geq \min\{\chi_I\ (x), \chi_I(y)\}$. So $\chi_I(xy) = 1 \geq \min\{\chi_I(x), \chi_I(y)\}$.

Then χ_I is a fuzzy subKS-semigroup .Now , since I is a subKS-semigroup and I is a non-empty subset of X thus $\exists \quad x \in I \quad such\ that \quad 0 = x * x \in I$. Therefore $\chi_I(0) = 1$. Hence χ_I is a normal fuzzy subKS-semigroup .Conversely ,suppose that χ_I is a normal fuzzy subKS-semigroup ,so χ_I is a fuzzy subKS-semigroup let $x, y \in I$ .To prove $x * y \in I$ since

$\chi_I(x * y) \geq \min\{\chi_I(x), \chi_I(y)\}$

$\chi_I(x) = 1, \chi_I(y) = 1$.So $\chi_I(x * y) \geq 1 \Rightarrow \chi_I(x * y) = 1$.

So $x*y\in I$ .Now ,let $x,y\in I$ so $\chi_I(x)=1,\chi_I(y)=1$ .Then $\chi_I(xy)\geq\min\{\chi_I(x),\chi_I(y)\}=1$,we get $xy\in I$.

4. Fuzzy Normal SubKS- Semigroup of KS- Semigroup

In this section we give the definition of fuzzy normal subKS-semigroup of X with related some properties ,like union , intersection , Cartesian product , strong , image and other properties of fuzzy subKS-semigroup .

Definition (4.1)

Let X be a KS-semigroups and μ a fuzzy set on X . Then μ is called a **fuzzy normal subKS-semigroup** of X if it satisfies the following conditions:

1) μ is a fuzzy subKS-semigroup of X .
2) $\mu(x*y)=\mu(y*x) \qquad \forall x,y\in X\setminus\{0\}$.
3) $\mu(xy)=\mu(yx) \qquad \forall x,y\in X$.

Remark(4.2)

The notions of fuzzy normal subKS-semigroup appear in 2011 such that $\mu(x*y)=\mu(y*x)$, $\forall x,y\in X$ since we find this condition imply to be μ constant since

$\mu(0*y)=\mu(y*0)\Rightarrow\mu(0)=\mu(y)$, $\forall y\in X$

then the results will be simple therefore we omitted{0} in our definition for the second condition .

Example (4.3)

Let X = {0, 1, 2 } be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Define a fuzzy subset $\mu:X\rightarrow[0,1]$ by $\mu(0)=0.6$ *and* $\mu(x)=0.2$ $\forall(x\neq 0)\in X$ then μ is a fuzzy normal subKS- semigroup of X .

Proposition (4.4)

Let μ and ν be fuzzy normal subKS-semigroups of X then $\mu\cap\nu$ be a fuzzy normal sub KS-semigroup .

Proof:

Let μ and ν be fuzzy normal subKS-semigroups of X .

Then $\mu\cap\nu$ is a fuzzy subKS-semigroup of X by[proposition 2.20]. Now,

$(\mu\cap\nu)(xy)=\min\{\mu(xy),\nu(xy)\}$

$=\min\{\mu(yx),\nu(yx)\}$ $[\mu,\nu$ *are fuzzy normal subKS – semigroups*$]$so,

$=(\mu\cap\nu)(yx)$, $\forall\ x,y\in X.$

$(\mu\cap\nu)(x*y)=\min\{\mu(x*y),\nu(x*y)\}$

$=\min\{\mu(y*x),\nu(y*x)\}[\mu,\nu$ *are fuzzy normal subKS – semigroups*$]$

$=(\mu\cap\nu)(y*x) \qquad \forall\ x,y\in X\setminus\{0\}.$

Therefore $\mu\cap\nu$ is a fuzzy normal subKS-semigroup **.**

Proposition (4.5)

Let μ and ν be fuzzy normal subKS-semigroup of X such that $\mu\subseteq\nu$ *or* $\nu\subseteq\mu$. Then $\mu\cup\nu$ be a fuzzy normal subKS-semigroup .

Proof:

Suppose that μ and ν be fuzzy normal subKS-semigroups,

then μ and ν are fuzzy sub KS-semigroups then
$\mu \cup \nu$ be a fuzzy subKS-semigroups [proposition 2.21]. Now,

$$(\mu \cup \nu)(xy) = \max\{\mu(xy), \nu(xy)\} = \max\{\mu(yx), \nu(yx)\} \quad [by\ hypothesis]$$
$$= (\mu \cup \nu)(yx) \quad \forall\ x, y \in X.$$

so,

$$(\mu \cup \nu)(x * y) = \max\{\mu(x * y), \nu(x * y)\} = \max\{\mu(y * x), \nu(y * x)\} \quad [by\ hypothesis]$$
$$= (\mu \cup \nu)(y * x) \quad \forall\ x, y \in X \setminus \{0\}.$$

Hence $\mu \cup \nu$ is a fuzzy normal subKS-semigroup .

Proposition (4.6)

Let λ and μ be fuzzy normal subKS-semigroups of X then $\lambda \times \mu$ is a fuzzy normal subKS-semigroup of $X \times X$.

Proof:

Let λ and μ be a fuzzy normal subKS-semigroups of X and let $(x_1, x_2), (y_1, y_2) \in X \times X$ *where* $x_1, x_2, y_1, y_2 \in X \ni x = (x_1, x_2)$, $y = (y_1, y_2)$ then λ and μ be a fuzzy subKS-semigroups of X so $\lambda \times \mu$ is a fuzzy subKS-semigroup [proposition 2.24].

$$(\lambda \times \mu)(xy) = (\lambda \times \mu)((x_1, x_2) \cdot (y_1, y_2)) = (\lambda \times \mu)(x_1 y_1, x_2 y_2) = \min\{\lambda(x_1 y_1), \mu(x_2 y_2)\}$$
$$= \min\{\lambda(y_1 x_1), \mu(y_2 x_2)\} \quad [\lambda, \mu\ are\ fuzzy\ normal\ subKS-semigroups]$$
$$= (\lambda \times \mu)((y_1, y_2) \cdot (x_1, x_2)) = (\lambda \times \mu)(yx)$$

and so ,

let $(x_1, x_2), (y_1, y_2) \in X \times X$ *where* $x_1, x_2, y_1, y_2 \in X \setminus \{0\}$ *such that*
$x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$

$$(\lambda \times \mu)(x * y) = (\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) = (\lambda \times \mu)(x_1 * y_1, x_2 * y_2) = \min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}$$
$$= \min\{\lambda(y_1 * x_1), \mu(y_2 * x_2)\}\ [\lambda, \mu\ are\ fuzzy\ normal\ subKS-semigroups]$$
$$= (\lambda \times \mu)((y_1, y_2) * (x_1, x_2)) = (\lambda \times \mu)(y * x).$$

Therefore $\lambda \times \mu$ is a fuzzy normal subKS-semigroup .

Proposition (4.7)

Let $\mu \times \lambda$ be a fuzzy normal subKS-semigroup of X.Then either λ or μ is a fuzzy normal subKS-semigroup of X .

Proof:

Let $\mu \times \lambda$ be a fuzzy normal subKS-semigroup of X .Then $\mu \times \lambda$ be a fuzzy subKS-semigroup of X then by [theorem 2.28] ,either λ or μ is a fuzzy subKS-semigroup of X if λ be a fuzzy subKS-semigroup of X so by [theorem 2.28] we have $\lambda(x) \le \mu(0)$ to prove λ is a normal let $x_1, x_2 \in X$ .Then

$$\lambda(x_1 x_2) = \min\{\mu(0), \lambda(x_1 x_2)\} = (\mu \times \lambda)(0, x_1 x_2) = (\mu \times \lambda)((0, x_1) \cdot (0, x_2))$$
$$= (\mu \times \lambda)((0, x_2) \cdot (0, x_1)) = (\mu \times \lambda)(0, x_2 x_1) = \min\{\mu(0), \lambda(x_2 x_1)\} = \lambda(x_2 x_1)$$

Now , let $x_1, x_2 \in X / \{0\}$

$$\lambda(x_1 * x_2) = \min\{\mu(0), \lambda(x_1 * x_2)\} = (\mu \times \lambda)(0, x_1 * x_2) = (\mu \times \lambda)((0, x_1) * (0, x_2))$$
$$= (\mu \times \lambda)((0, x_2) * (0, x_1)) = (\mu \times \lambda)(0, x_{2*} x_1) = \min\{\mu(0), \lambda(x_2 * x_1)\} = \lambda(x_{2*} x_1)$$

Hence λ is a fuzzy normal subKS-semigroup .In a similer way if $\mu\times\lambda$ is a fuzzy normal subKS-semigroup and μ is a fuzzy subKS-semigroup .We can prove that μ is a fuzzy normal subKS-semigroup .

Proposition (4.8)

Let A be a subKS-semigroup and let μ defined by

$\mu(x)=\begin{cases} t_1 & if \quad x\in A \\ t_2 & if \quad x\notin A \end{cases}$ $, where \quad t_1 < t_2$. Then μ is a fuzzy normal sub KS-semigroup .

Proof:

Let A be a subKS-semigroup and let μ defined by

$$\mu(x)=\begin{cases} t_1 & if \quad x\in A \\ t_2 & if \quad x\notin A \end{cases} \quad , where \quad t_1 < t_2$$

In first we prove μ is a fuzzy subKS-semigroup of X ,let $x, y\in X$. Then

1) $If \quad x,y\in A \quad then \quad xy\in A\rightarrow \mu(xy)=t_1 \geq \min\{\mu(x),\mu(y)\}$

2) $If \quad x,y\notin A \quad then \quad xy\notin A\rightarrow \mu(xy)=t_2 \geq \min\{\mu(x),\mu(y)$

3) $If\ x\in A, y\notin A\ then\ \mu(x)=t_1,\ \mu(y)=t_2\ and\ (xy)\notin A\ so\ \mu(xy)=t_2 \geq \min\{\mu(x),\mu(y)\}$

$=\min\{t_1, t_2\ \}=t_1$

4) $If\ x\notin A, y\in A\ then\ \mu(x)=t_2\ ,\mu(y)=t_1\ and\ (xy)\notin A\ so\ \mu(xy)=t_2 \geq \min\{\mu(x),\mu(y)\}$

$=\min\{t_2, t_1\}=t_1$

so $\mu(xy)\geq \min\{\mu(x),\mu(y)\}$ by a similar way $\mu(x*y)\geq \min\{\mu(x),\mu(y)\}\ \forall x,y\in X\setminus\{0\}$. Therefore , μ is a fuzzy subKS-semigroup .Now,

1) $if\ \ x,y\in A$ since $A\ is\ subKS-semigrroup \Rightarrow xy\in A, yx\in A\ ,then\ \ \mu(xy)=t_1\ =\mu(yx)$.

2) $if\ \ x,y\notin A$ since $A is\ subKS-semigrroup \Rightarrow xy\notin A, yx\notin A\ ,then\ \ \mu(xy)=t_2\ =\mu(yx)$.

3) $if\ \ x\in A, y\notin A\ \ then\ \ \ xy\notin A, yx\notin A\ \ .\ Thus\ \ \ \mu(xy)=t_2\ =\mu(yx)$.

4) $if\ \ x\notin A, y\in A\ \ then\ \ \ xy\notin A, yx\notin A\ \ .Thus\ \ \ \mu(xy)=t_2\ =\mu(yx)$.

so $\mu(xy)=\mu(yx) \quad for\ all\ \ x,y\in X$. Also,

1) $If\ \ x,y\in A$ [since $A\ is\ subKS-semigrroup \Rightarrow x*y\in A, y*x\in A$]
 $then\ \ \mu(x*y)=t_1\ =\mu(y*x)$.

2) $If\ \ x,y\notin A$ [since $A\ is\ subKS-semigrroup \Rightarrow x*y\notin A, y*x\notin A$]
 $then\ \ \mu(x*y)=t_2\ =\mu(y*x)$.

3 $If\ x\in A, y\notin A\ \ \ then\ \ \ x*y\notin A, y*x\notin A\ \ .\ Thus\ \ \mu(x*y)=t_2\ =\mu(y*x)$.

4) $If\ \ x\notin A, y\in A\ \ \ then\ \ \ x*y\notin A, y*x\notin A\ \ \ .Thus\ \ \mu(x*y)=t_2\ =\mu(y*x)$.

so $\mu(x*y)=\mu(y*x) \quad for\ all\ \ x,y\in X\setminus\{0\}$.Hence μ is a fuzzy normal subKS-semigroup .

Proposition (4.9)

Let ν be a fuzzy subKS-semigroup of X and ρ_ν strongest fuzzy relation on X. Then ν is a fuzzy normal subKS-semigroup if and only if ρ_ν is a fuzzy normal subKS-semigroup of $X\times X$.

Proof:

Let ν be a fuzzy normal subKS-semigroup of X $, x=(x_1, x_2)\ \ and\ \ y=(y_1, y_2)\in X\times X$. Now clearly ρ_V is a fuzzy subKS-semigroup by [proposition 2.27] . Now,

$$\rho_V(xy) = \rho_V(x_1y_1, x_2y_2) = \min\{\nu(x_1y_1), \nu(x_2y_2)\}$$
$$= \min\{\nu(y_1x_1), \nu(y_2x_2)\} \quad [\nu \text{ is a fuzzy normal subKS} - \text{semigroup}]$$
$$= \rho_V(y_1x_1, y_2x_2)$$
$$= \rho_V((y_1, y_2)\cdot(x_1, x_2)) = \rho_V(yx)$$

and , let $x, y \in X \setminus \{0\}$

$$\rho_V(x*y) = \rho_V(x_1*y_1, x_2*y_2) = \min\{\nu(x_1*y_1), \nu(x_2*y_2)\}$$
$$= \min\{\nu(y_1*x_1), \nu(y_2*x_2)\} \; [\nu \text{ is a fuzzy normal subK} - \text{semigroup}]$$
$$= \rho_V(y_1*x_1, y_2*x_2) = \rho_V((y_1, y2)*(x_1, x_2)) = \rho_V(y*x)$$

Hence ν is a fuzzy normal subKS-semigroup .Conversely , assume that ρ_V is a fuzzy normal subKS semigroup of $X \times X$ then for any $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in X \times X$. We know that ν is a fuzzy subKS-semigroup. To prove that ν is a fuzzy normal subKS-semigroup there are two cases :

1) *If* $\nu(x_1y_1) \le \nu(x_2y_2)$ *where* $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$. *Then*

$$\nu(x_1y_1) = \min\{\nu(x_1y_1), \nu(x_2y_2)\} = \rho_V(xy) = \rho_V(yx) = \rho_V\left[(y_1, y_2)\cdot(x_1, x_2)\right] = \rho_V(y_1x_1, y_2x_2)$$
$$= \min\left[\nu(y_1x_1), \nu(y_2x_2)\right] = \nu(y_1x_1)$$

since *if* $\nu(x_1y_1) = \nu(y_2x_2)$ *, we have*

$\nu(x_1y_1) < \nu(x_2y_2)$ *and* $\nu(y_2x_2) < \nu(y_1x_1)$

so $\nu(x_1y_1) < \min\{\nu(x_2y_2), \nu(y_1x_1)\} = \rho_V((x_2y_2), (y_1x_1))$

then $\nu(x_1y_1) < \rho_V((x_2y_1).(y_2x_1)) = \rho_V((y_2x_1).(x_2y_1)) = \rho_V(y_2x_2, x_1y_1)$

$$= \min\{\nu(y_2x_2), \nu(x_1y_1)\} = \nu(x_1y_1)$$

where $(x_2y_1), (y_2x_1) \in X \times X$ since $x_1, x_2, y_1, y_2 \in X$

so $\nu(x_1y_1) < \nu(x_1y_1)$ *. This contraduction*

we get $\nu(x_1y_1) = \nu(y_1x_1) \quad \forall (x_1, x_2), (y_1, y_2) \in X \times X.$

2) *If* $\nu(x_2y_2) \le \nu(x_1y_1)$

In a similar way we get $\nu(x_2y_2) = \nu(y_2x_2) \quad \forall (x_1, x_2), (y_1, y_2) \in X \times X$

Now , let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$ such that $x_1, x_2, y_1, y_2 \in X \setminus \{0\}$ there are two cases :

1) *If* $\nu(x_1*y_1) \le \nu(x_2*y_2)$ *then*

$$\nu(x_1*y_1) = \min\{\nu(x_1*y_1), \nu(x_2*y_2)\} = \rho_V(x*y) = \rho_V(y*x)$$
$$= \rho_V\left[(y_1, y_2)*(x_1, x_2)\right] = \min\left[\nu(y_1*x_1), \nu(y_2*x_2)\right] = \nu(y_1*x_1).$$

so $\nu(x_1*y_1) = \nu(y_1*x_1) \quad \forall x_1, x_2, y_1, y_2 \in X \setminus \{0\}$

2) *If* $\nu(x_2*y_2) \le \nu(x_1*y_1)$ .In similar way we get $\nu(x_2*y_2) = \nu(y_2*x_2)$.

Hence ν is a fuzzy normal subKS-semigroup .

Remark(4.10)

If μ is a fuzzy set such that for all $x \in X$, $\mu(x) = 1$ then μ is a normal fuzzy subKS-semigroup and μ is a fuzzy normal subKS-semigroup .

proof: it is clear

Proposition (4.11)

Let $f : X \to Y$ be a homomorphism if μ is a fuzzy normal subKS-semigroup of Y . Then μ^f is a fuzzy normal subKS-semigroup of X .

Proof:

Let μ be a fuzzy normal subKS-semigroup of Y .Then

μ^f is a fuzzy subKS-semigroup [by Proposition (3.7)].Now , let $x, y \in X$,so

$$\mu^f(xy) = \mu(f(xy)) = \mu(f(x) \cdot f(y)) \quad [f \text{ is a homomorphism}]$$
$$= \mu(f(y) \cdot f(x)) \quad [\text{since } \mu \textit{ is fuzzy normal subKS} - \textit{semigroup}]$$
$$= \mu(f(yx) = \mu^f(yx)$$

and let $x, y \in X/\{0\}$, so

$$\mu^f(x * y) = \mu(f(x * y)) \quad = \mu(f(x) * f(y)) \quad [f \text{ is a homomorphism}]$$
$$= \mu(f(y) * f(x)) \quad [\text{since } \mu \textit{ is fuzzy normal subKS} - \textit{semigroup}]$$
$$= \mu(f(y * x)) = \mu^f(y * x).$$

Hence μ^f is a fuzzy normal subKS-semigroup .

Proposition (4.12)

Let $f : X \to Y$ be an epimorphism if μ^f is a fuzzy normal subKS-semigroup of X . Then μ is a fuzzy normal subKS-semigroup of Y .

Proof:

Let μ^f be a fuzzy normal subKS-semigroup of X so μ is a fuzzy subKS-semigroup [by Proposition (3.8)]

let $x, y \in Y$ $\exists$ $a, b \in X$ *such that* $f(a) = x$, $f(b) = y$ [f is an epimorphism]

$$\mu(xy) = \mu(f(a) \cdot f(b)) = \mu(f(ab)) = \mu^f(ab) = \mu^f(ba) \quad [\text{since } \mu^f \textit{ is a fuzzy normal subKS} - \textit{semigroup}]$$
$$= \mu(f(ba)) = \mu(f(b) \cdot f(a)) = \mu(yx).$$

and let $x, y \in Y/\{0\}$, *so* $\exists$ $a, b \in X$ *such that* $f(a) = x$, $f(b) = y,$ so $a, b \neq 0$ since $f(0) = 0$ in a similar way we can prove $\mu(x * y) = \mu(y * x)$.

Hence μ is a fuzzy normal subKS-semigroup .

Proposition (4.13)

Let μ be a non constant fuzzy normal subKS-semigroup of X such that μ is a fuzzy KS-ideal of X then

$$\mu = \begin{Bmatrix} \mu(0) & \textit{if} & x = 0 \\ t & \textit{if} & x \neq 0 \end{Bmatrix} \quad , \textit{where} \quad t \in [0, \mu(0))$$

Proof:

Let μ be a non constant fuzzy normal subKS-semigroup of X,

and let $x, y \in X/\{0\}$, since μ is a fuzzy KS-ideal ,so

$\mu(x) \geq \min\{\mu(x*y), \mu(y)\} = \min\{\mu(y*x), \mu(y)\}$ [$since\ \mu\ is\ a\ fuzzy\ normal\ subKS-semigroup$]

$but\quad \mu(y*x) \geq \mu(y)\quad .Hence\quad \mu(x) \geq \mu(y),$

$but\quad \mu(x) \geq \mu(y) \geq \min\{\mu(y*x), \mu(x)\}\quad [by\ definition\ of\ KS-ideal]$

$$= \min\{\mu(x*y), \mu(x)\} = \mu(x)$$

since $\mu(x*y) \geq \mu(x)$ $implies$ $\mu(y) = \mu(x)$. $for\ all$ $x, y \in X$ $such\ that$ $x, y \neq 0$

that is mean μ is a constant for each $x, y \neq 0$

$since$ $\mu(0) \geq \mu(x)$ $for\ all$ $x \in X$, and μ is a non constant, so

$$\mu = \begin{cases} \mu(0) & if \quad x = 0 \\ t & if \quad x \neq 0 \end{cases} \quad where \quad t \in [0, \mu(0)) .$$

Proposition (4.14)

Let μ be a fuzzy normal subKS-semigroup of X .Then the fuzzy set μ^{+} defined by $\mu^{+}(x) = \mu(x) + 1 - \mu(0)$ is a fuzzy normal subKS-semigroup .

Proof:it is clear

Proposition (4.15)

Let I be a non-empty subset of X . Then I is a subKS-semigroup if only if χ_I is a fuzzy normal subKS-semigroup, where

$$\chi_I = \begin{cases} 1 & if \quad x \in I \\ 0 & if \quad x \notin I \end{cases}$$

Proof:

Let I is a subKS-semigroup of X.Then

χ_I is a fuzzy subKS-semigroup [by Proposition (3.18)]

Now, $x, y \in X$ then there are several cases :

1) $If\ x, y \in I\ then\ xy \in I\ and\ yx \in I\ then\ \chi_I(xy) = \chi_I(yx) = 1.$

2) $If\ x, y \notin I\ then\ xy \notin I\ and\ yx \notin I\ then\ \chi_I(xy) = \chi_I(yx) = 0.$

3) $If\ x \in I, y \notin I\ then\ xy, yx \notin I\ then\ \chi_I(xy) = \chi_I(yx) = 0.$

4) $If\ x \notin I, y \in I\ then\ xy, yx \notin I\ then\ \chi_I(xy) = \chi_I(yx) = 0.$

$thus\ \chi_I(xy) = \chi_I(yx).$

In a similar way we can prove $\chi_I(x*y) = \chi_I(y*x)\quad \forall\ x, y \in X \setminus \{0\}.$

Hence χ_I is a fuzzy normal subKS-semigroup .

Conversely ,

let χ_I is a fuzzy normal subKS-semigroup we can prove I is a subKS-semigroup of X in a similar way of Proposition (3.18) .

Proposition (4.16)

Let X be a KS-semigroup and $a, b \in X \setminus \{0\}$ $such\ that$ $a^2 = a,\ b^2 = b$ and $ax = a\ , bx = b\ \ \forall\ x \in X \setminus \{0\}$. $Then$ $\mu(a) = \mu(b)$ where μ is a fuzzy normal subKS-semigroup .

Proof:

Let μ be a fuzzy normal subKS-semigroup of X and

Let $a, b \in X \setminus \{0\}$ $such\ that$ $a^2 = a,\ b^2 = b$ .Then

$\mu(a) = \mu(ax)\quad [a = ax\ \ \forall\ x \in X \setminus \{0\}]\ \ let\ x = b$

$\mu(a) = \mu(ab) = \mu(ba)\ [\mu\ is\ a\ normal\ fuzzy\ subKS-semigroup]$

$so\ \ \mu(a) = \mu(b)\ \ [since\ bx = b\ \ \forall\ x \in X \setminus \{0\}\ ]$.

5 . Fuzzy Normal SubKS- Semigroup of Fuzzy SubKS- Semigroup

In this section , we give the definition of fuzzy normal subKS-semigroup of fuzzy subKS-semigroup of X and give some of properties ,like union , intersection , Cartesian product ,image and other properties of fuzzy subKS-semigroup of fuzzy subKS-semigroup .

Definition (5.1)

Let X be a KS-semigroups , μ and ν are fuzzy subKS-semigroups of X such that $\mu \subseteq \nu$ then μ is called **fuzzy normal subKS-semigroup of fuzzy subKS-semigroup** ν if :

(1) $\mu(y*x) \geq \min\{\mu(x*y), \nu(y)\}$

(2) $\mu(yx) \geq \min\{\mu(xy), \nu(y)\}$, $\forall$ $x, y \in X$.

Example (5.2)

Let X = {0, 1, 2,3 } be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	0
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Define two fuzzy sets μ , ν $: X \rightarrow [0,1]$, by

$\nu(0) = 0.8$, $\nu(1) = 0.4$, $\nu(2) = 0.2$, $\nu(3) = 0.1$ and

$\mu(0) = 0.5$, $\mu(1) = 0.4$, $\mu(2) = 0.2$, $\mu(3) = 0.1$ then by usual calculations we can prove that μ is a fuzzy normal subKS- semigroup of ν.

Proposition (5.3)

Let X be a KS-semigroup and let μ and λ be fuzzy normal subKS-semigroup of fuzzy subKS-semigroups ν . Then $\mu \cap \lambda$ is a fuzzy normal sub KS-semigroup of ν .

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν .

then $\mu \cap \lambda$ is a fuzzy subKS-semigroup [proposition 2.20].

Now , let $x, y \in X$, since $\mu(y*x) \geq \min\{\mu(x*y), \nu(y)\}$, $\lambda(y*x) \geq \min\{\lambda(x*y), \nu(y)\}$ and $\mu(yx) \geq \min\{\mu(xy), \nu(y)\}$, $\lambda(yx) \geq \min\{\lambda(xy), \nu(y)\}$ .Therefore

1) $(\mu \cap \lambda)(yx) = \min\{\mu(yx), \lambda(yx)\} \geq \min\{\min\{\mu(xy), \nu(y)\}, \min\{\lambda(xy), \nu(y)\}\}$

$= \min\{\min\{\mu(xy), \lambda(xy)\}, \min\{\nu(y), \nu(y)\} = \min\{(\mu \cap \lambda)(xy), \nu(y)\}$

and,

2) $(\mu \cap \lambda)(y*x) = \min\{\mu(y*x), \lambda(y*x)\} \geq \min\{\min\{\mu(x*y), \nu(y)\}, \min\{\lambda(x*y), \nu(y)\}\}$

$= \min\{\min\{\mu(x*y), \lambda(x*y)\}, \min\{\nu(y), \nu(y)\} = \min\{(\mu \cap \lambda)(x*y), \nu(y)\}$

Hence $\mu \cap \lambda$ is a fuzzy normal subKS-semigroup of ν .

Proposition (5.4)

Let X be a KS-semigroups and let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν then $\mu \cup \lambda$ is a fuzzy normal subKS-semigroup of ν if $\mu \subseteq \lambda$ *or* $\lambda \subseteq \mu$.

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν . $\mu \cup \lambda$ is a fuzzy subKS-semigroup so by [proposition 2.21] . Now , let $x, y \in X$ .Then

1) $(\mu \cup \lambda)(yx) = \max\{\mu(yx), \lambda(yx)\} \geq \max\{\min\{\mu(xy), \nu(y)\}, \min\{\lambda(xy), \nu(y)\}\}$

$= \min\{\max\{\mu(xy), \lambda(xy)\}, \max\{\nu(y), \nu(y)\}$ [since $\mu \subseteq \lambda$ *or* $\lambda \subseteq \mu$]

$= \min\{(\mu \cup \lambda)(xy), \nu(y)\}$

and so ,

2) $(\mu \cup \lambda)(y * x) = \max\{\mu(y * x), \lambda(y * x)\}$

$\geq \max\{\min\{\mu(x * y), \nu(y)\}, \min\{\lambda(x * y), \nu(y)\}\}$

$= \min\{\max\{\mu(x * y), \lambda(x * y)\}, \max\{\nu(y), \nu(y)\}$ [$\mu \subseteq \lambda$ *or* $\lambda \subseteq \mu$]

$= \min\{(\mu \cup \lambda)(x * y), \nu(y)\}$

Hence $\mu \cup \lambda$ is a fuzzy normal subKS-semigroup of ν .

Proposition (5.5)

Let A, ν be fuzzy subKS-semigroup in X and ρ_ν strongest fuzzy relation on X if A is a fuzzy normal subKS-semigroup of ν . Then ρ_A is a fuzzy normal subKS-semigroup of ρ_ν .

Proof:

Let A be a fuzzy normal subKS-semigroup of ν so A is a fuzzy subKS-semigroup of X so by [proposition 2.27] ρ_A is a fuzzy subKS-semigroup of X and ρ_ν is a fuzzy subKS-semigroup of X Now, to prove ρ_A be a fuzzy normal subKS-semigroup of ρ_ν .

1) $\rho_A(yx) = \rho_A(y_1 x_1, y_2 x_2) = \min\{A(y_1 x_1), A(y_2 x_2)\}$

$\geq \min\{\min\{A(x_1 y_1), \nu(y_1)\}, \min\{A(x_2 y_2), \nu(y_2)\}\}$ [*A is a fuzzy normal of* ν]

$= \min\{\min\{A(x_1 y_1), A(x_2 y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} = \min\{\rho_A(x_1 y_1, x_2 y_2), \rho_\nu(y)\}$

$= \min\{\rho_A(xy), \rho_\nu(y)\}$

2) $\rho_A(y * x) = \rho_A(y_1 * x_1, y_2 * x_2) = \min\{A(y_1 * x_1), A(y2 * x_2)\}$

$\geq \min\{\min\{A(x_1 * y_1), \nu(y_1)\}, \min\{A(x_2 * y_2), \nu(y_2)\}\}$ [*A is afuzzy normal of* ν]

$= \min\{\min\{A(x_1 * y_1), A(x_2 * y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\}$

$= \min\{\rho_A(x_1 * y_1, x_2 * y_2), \rho_\nu(y)\} = \min\{\rho_A(x * y), \rho_\nu(y)\}.$

Hence ρ_A is a fuzzy normal subKS-semigroup of ρ_ν .

Proposition (5.6)

If μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν then $\mu \times \lambda$ is a fuzzy normal subKS-semigroup of $\nu \times \nu$.

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of ν .

let $(x_1, x_2), (y_1, y_2) \in X \times X$ *such that* $x = (x_1, x_2)$, $y = (y_1, y_2)$

so ν, μ, λ are fuzzy subKS-semigroup of X , then by [proposition 2.24] $\nu \times \nu$ is a fuzzy subKS-semigroup , so by [proposition 2.24] then $\mu \times \lambda$ is a fuzzy subKS-semigroup of $X \times X$. Now, to prove $\mu \times \lambda$ is a fuzzy normal subKS-semigroup of $\nu \times \nu$

$(\mu \times \lambda)(yx) = (\mu \times \lambda)((y_1, y_2)(x_1, x_2)) = (\mu \times \lambda)(y_1 x_1, y_2 x_2) = \min\{\mu(y_1 x_1), \lambda(y_2 x_2)\}$

$\geq \min\{\min\{\mu(x_1 y_1), \nu(y_1)\}, \min\{\lambda(x_2 y_2), \nu(y_2)\}\}$

$= \min\{\min\{\mu(x_1 y_1), \lambda(x_2 y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\}$

$= \min\{(\mu \times \lambda)((x_1, x_2)(y_1, y_2)), \nu \times \nu(y_1, y_2)\} = \min\{(\mu \times \lambda)(xy), \nu \times \nu(y)\}$

and so ,

$(\mu\times\lambda)(y*x)=(\mu\times\lambda)((y_1,y_2)*(x_1,x_2))=(\mu\times\lambda)(y_1*x_1,y_2*x_2)$

$=\min\{\mu(y_1*x_1),\lambda(y_2*x_2)\}\geq\min\{\min\{\mu(x_1*y_1),\nu(y_1)\},\min\{\lambda(x_2*y_2),\nu(y_2)\}\}$

$=\min\{\min\{\mu(x_1*y_1),\lambda(x_2*y_2)\},\min\{\nu(y_1),\nu(y_2)\}\}$

$=\min\{(\mu\times\lambda)((x_1,x_2)*(y_1,y_2)),\nu\times\nu(y_1,y_2)\}=\min\{(\mu\times\lambda)(x*y),\nu\times\nu(y)\}$

Hence $\mu\times\lambda$ is a fuzzy normal subKS-semigroup of $\nu\times\nu$.

Proposition (5.7)

Let $f:\mathrm{X}\rightarrow\mathrm{Y}$ be a homomorphism if μ is a fuzzy normal subKS-semigroup of a fuzzy subKS-semigroup ν . Then μ^f is a fuzzy normal subKS-semigroup of ν^f .

Proof:

Let μ is a fuzzy normal subKS-semigroup of ν . Then

$\mu^f(x)=\mu(f(x))\leq\nu(f(x))=\nu^f(x)$ [since $\mu\subseteq\nu$] and μ^f is a fuzzy subKS-semigroup where by Proposition (3.8) ν^f is a fuzzy subKS-semigroup .

Now, to prove μ^f is a fuzzy normal subKS-semigroup of ν^f . Since

$\mu^f(yx)=\mu(f(yx))=\mu(f(y)f(x))\geq\min\{\mu(f(x)f(y)),\nu(f(y))\}$

$=\min\{\mu(f(xy)),\nu^f(y)\}=\min\{\mu^f(xy),\nu^f(y)\}$

so,

$\mu^f(y*x)=\mu(f(y*x))=\mu(f(y)*f(x))\geq\min\{\mu(f(x)*f(y)),\nu(f(y))\}$

$=\min\{\mu(f(x*y)),\nu^f(y)\}=\min\{\mu^f(x*y),\nu^f(y)\}$

Hence μ^f is a fuzzy normal subKS-semigroup of ν^f .

Proposition (5.8)

Let $f:\mathrm{X}\rightarrow\mathrm{Y}$ epimorphism if μ^f is a fuzzy normal subKS-semigroup of ν^f . Then μ is a fuzzy normal subKS-semigroup of ν .

Proof:

Let μ^f is a fuzzy normal subKS-semigroup of ν^f then since f is an epimorphism if $x\in\mathrm{Y}\ \exists\, a\in\mathrm{X}$ *such that* $f(a)=x$

$\mu(x)=\mu(f(a))=\mu^f(a)\leq\nu^f(a)=\nu(f(a))=\nu(x)$ *so* $\mu(x)\subseteq\nu(x)\ \forall x\in\mathrm{Y}$ and μ is a fuzzy subKS-semigroup [by Proposition (3.8)]

Now , let $x,y\in\mathrm{Y}\ \exists\, a,b\in X$ *such that* $f(a)=x$, $f(b)=y$. Then

$\mu(yx)=\mu(f(b)f(a))=\mu(f(ba))=\mu^f(ba)\geq\min\{\mu^f(ab),\nu^f(b)\}$

$=\min\{\mu(f(ab)),\nu(f(b))\}=\min\{\mu(f(a)f(b)),\nu(y)\}=\min\{\mu(xy),\nu(y)\}$

and so ,

$\mu(y*x)=\mu(f(b)*f(a))=\mu(f(b*a))=\mu^f(b*a)\geq\min\{\mu^f(a*b),\nu^f(b)\}$

$=\min\{\mu(f(a*b)),\nu(f(b))\}=\min\{\mu(f(a)*f(b)),\nu(y)\}=\min\{\mu(x*y),\nu(y)\}.$

Hence μ is a fuzzy normal subKS-semigroup of ν.

Proposition (5.9)

Let μ be a fuzzy normal subKS-semigroup of ν and μ is a normal fuzzy subKS-semigroup of X . Then $\mu=\nu$.

Proof:

let μ be a fuzzy normal subKS-semigroup of ν , so $\mu \subseteq \nu$ since $\mu(y*x) \geq \min\{\mu(x*y), \nu(y)\}$ since μ is a normal fuzzy subKS-semigroup of X so $\mu(0)=1$ .Now, *if* $x=0$ we have $\mu(y) \geq \min\{\mu(0), \nu(y)\} = \min\{1, \nu(y)\} = \nu(y)$. But $\mu(y) \leq \nu(y)$, $\forall\ y \in X$ *so* $\mu(y) = \nu(y)$, $\forall\ y \in X$

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