Stability for hybrid mapping of quasi-contraction in b-metric space

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Abstract:

In this paper, we prove the stability of iterative process for hybrid mappings of quasi-contraction in b-metric space. Our work generalizes and improves some results due to Czerwik*et al.* 1997, Singh*et al.* 2008 and Hashim 2011.

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1. Introduction

The first result on stability of iteration on metric spaces was given by M.Ostrowski 1956, although this type of problem was already observed M. Urabe in 1957 for real valued function. Thereafter, a number of authors have generalized and extended his result in various setting. Harder and Hicks have obtain interesting results on stability of a number of iterative procedures with examples. Rhoades (1990) and Rhoades and Saliga (2001)have a generalized various existing result and established fixed point theorems for various iterative procedures. Czerwik 1997 and Czerwik et al. 2002 has extended Ostrroski's classical theorem for the stability of iterative procedure for multi-valued maps in b-metric space .in 2011 HashimA.M. studied the stability of iterative process for hybrid contraction mappings involving a single-valued and multi-valued maps in b-metric space.

The purpose of this paper is to study the stability of iterative process for hybrid maps satisfying quasi-contraction in b-metric space.

2. PRELIMINARIES

Definition (2.1) (Czerwik1998)

Let X be a nonempty set and let $b \geq 1$ be a given real number . A function $d : X \times X \rightarrow R^{+}$ (The set of positive real numbers) is called a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x,y) = 0 \; iff \; x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

The pair (X, d) is called a b-metric space.

It is clear that definition of b-metric space is an extension of usual metric space. The following example illustrates that b-metric space are more general than metric spaces.

Example (2.2)(Singh *et al.* 2008)

Let $X = \{x_1, x_2, x_3, x_4\}$ and

$d(x_1,x_2) = b \geq 2, d(x_1,x_3) = d(x_1,x_4) = d(x_2,x_3) = d(x_2,x_4) = d(x_3,x_4) = 1$

$d(x_{i,} x_j) = d(x_j, x_i)$ for $i, j = 1,2,3,4$ and $d(x_i, x_i) = 0, i = 1,2,3,4.$

$d(x_i, x_j) \leq \frac{b}{2}[d(x_i, x_n) + d(x_n, x_j)]$ for i,j=1,2,3,4, andif $b > 2$, the ordinary triangle inequality does not hold.i.e. need not be a metric on X

Let (X,d) is a b-metric space.

$CL(X)=\{A : A \text{ is nonempty closed subset of } X\}$
$CB(X)=\{A : A \text{ is nonempty closed bounded subset of } X\}$

$$H(A,B)=Max\begin{Bmatrix} \sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A), & \text{if the maximum exists} \\ \infty, & \text{otherwise} \end{Bmatrix},$$

$A,B\in CL(X)$, $D(x,A)=\inf_{a\in A} d(x,a)$

H is called a generalized Hausdorff b-metric on $CL(X)$ induced by the metric d on X.

We shall require the following Lemmas and definitions

Lemma (2.3)(Czerwik 1998)

Let (X,d) be a b-metric space, then for any $A,B,C\in CL(X)$,

1) $D(x,B)\le d(x,y)$ for any $y\in B$,
2) $D(A,B)\le H(A,B)$,
3) $H(A,C)\le b[H(A,B)+H(B,C)]$.

Lemma(2.4) (Czerwik 1998)

Let A and B be distinct element of $CL(X)$. Let $x\in A$. Then for a given number $\lambda>1$, there exists a point y in B such that $d(x,y)\le\lambda H(A,B)$.

Definition (2.5) (Singh *et al*. 2008)

Let (X,d) be a b-metric space and $Y\subseteq X$. Let $T:Y\to CL(X)$, $S:Y\to X$, $T(Y)\subseteq S(X)$. and z a coincidence point of T and S, that is $Sz\in Tz=u$. For any $x_0\in Y$, let the sequence $\{Sx_n\}$ generated by the general iterative procedure.

$Sx_{n+1}\in f(T,x_n)$ $\quad n=1,2,\dots$.
converge to an element $u\in X$. Let $\{Sy_n\}\subseteq X$ be an arbitrary sequence and let $\varepsilon_n=H(y_{n+1},f(T,y_n))$, $n=0,1,2,\dots,$. Then the iterative procedure $f(T,x_n)$ is (S,T)-stable, or stable with respect

to (S,T) if and only if $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty} Sy_n = u$.

This definition in metric spaces with $b = 1$ and (The identity map) is due to Harder and Hick (1988a, 1988b) and Rhoades (1990,1993) and Rhoades and Saliga(2001) and Singh and Chadha (1995).

Definition (2.6) (Rao and Swamy in 2013)

Let (X,d) be a b-metric space. Let $T: X \to CB(X)$ be multivalued and $S: X \to X$ be a self-map. The hybrid pairs (T,S) are said to be q-multivalued quasi-contraction if there exists $0 \le q < 1$ such that

$$H(Tx,Ty) \le qM(x,y) \quad (2.1)$$

for all $x, y \in X$, where,

$$M(x,y) = \max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$$

Theorem (2.7) (Rao and Swamy in 2013)

Let (X,d) be a b-metric space. Suppose that $S: X \to X$ and $T: X \to CB(X)$.

Suppose that the hybrid pair (T,S) is q-multivalued quasi-contraction, $TX \subseteq SX$ and SX is complete. Assume that $q < \frac{1}{b^2+b}$. Then

i) T and S have coincidence points in X (or)

ii) The pairs (T,S) have a common coincidence point.

Lemma (2.8) see also (Rhoades 1993, Czerwik et al. 2002)

Let $\{\varepsilon_n\}$ be a sequence of non negative real numbers. Let $s_n = \sum_{i=0}^{n} k^{n-1}\varepsilon_i$ where $0 \le k < 1$. Then $\lim_{n\to\infty}\varepsilon_n = 0$ iff $\lim_{n\to\infty} s_n = 0$.

In this paper, we prove the stability of quasi-contraction for hybrid mappings

3.Main Results

Theorem (3.1)

Let (X,d) be a b-metric space. Suppose that SX or TX is a complete subspace of X and let $S: X \to X$ and $T: X \to CL(X)$ and suppose that the hybrid pairs (S,T) is q-multivalued quasi-contraction. Let z be a coincidence point of T and S i.e $z = Su \in Tu$. For any $x_0 \in X$ let the sequence $\{Sx_n\}$ generated by $Sx_{n+1} \in Tx_n$

converge to $z \in X$. Let $\{Sy_n\} \subseteq X$ be an arbitrary sequence and define

$$\varepsilon_n = H(Sy_{n+1}, Ty_n), \quad n = 0,1,2,..., \quad k = bq < 1 \; and \; q < \frac{1}{b^2 + b}, \text{ then}$$

1) $d(z, Sy_{n+1}) \le bd(z, Sx_{n+1}) + b(b\alpha)^{n+1} d(x_0, y_0) + b^3\alpha \sum_{r=0}^{n} (b\alpha)^{n-r} d(Sx_r, Tx_r) + b^2 \sum_{r=0}^{n} (b\alpha)^{n-r} \varepsilon_r$

2) If Tz is singleton then $\lim_{n\to\infty} y_n = z$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

Proof:

From (3.1) for any $x, y \in X$, one of the following holds:

$$H(Tx, Ty) \le qd(Sx, Sy),$$

$$H(Tx, Ty) \le q\, D(Sx, Tx)$$
$$\le q\, H(Sx, Tx);$$

$$H(Tx, Ty) \le q\, D(Sy, Ty)$$
$$\le q\, b[H(Sy, Sx) + H(Sx, Ty)]$$

$$\le qb[d(Sy, Sx)] + qb[bH(Sx, Tx) + bH(Tx, Ty)]$$

$$\le qb[d(Sy, Sx)] + qb^2 H(Sx, Tx) + qb^2 H(Tx, Ty)$$

i.e. $H(Tx, Ty) \le \frac{qb}{1 - qb^2} d(Sy, Sx) + \frac{qb^2}{1 - qb^2} H(Sx, Tx)$

$$\le \frac{k}{1 - kb} d(Sy, Sx) + \frac{kb}{1 - kb} H(Sx, Tx)$$

$$H(Tx, Ty) \le qd(Sx, Ty)$$
$$\le qb[d(Sx, Tx) + H(Tx, Ty)]$$

$$\leq \frac{qbH(Sx,Tx)}{(1-qb)}$$

$$\leq \frac{k}{1-k} H(Sx,Tx)$$

$$H(Tx,Ty) \leq qd(Sy,Tx)$$
$$\leq qb[d(Sy,Sx)+H(Sx,Tx)]$$

Therefore, in all cases we get,

$$H(Tx,Ty) \leq \alpha d(Sx,Sy)+b\alpha H(Sx,Ty), \text{ where } \alpha = \frac{b^2 q}{1-b^2 q}, \; b\alpha < 1$$

$$d(Sx_{n+1},Sy_{n+1}) \leq H(Tx_n,Sy_{n+1})$$
$$\leq b[H(Tx_n,Ty_n)+H(Ty_n,Sy_{n+1})]$$
$$\leq b[\alpha d(Sx_n,Sy_n)+b\alpha H(Sx_n,Ty_n)+\varepsilon_n]$$
$$\leq b\alpha\, d(Sx_n,Sy_n)+b^2\alpha H(Sx_n,Ty_n)+b\varepsilon_n$$
$$\leq b^2\alpha[H(Tx_{n-1},Ty_{n-1})+H(Ty_{n-1},Sy_n)]+b^2\alpha H(Sx_n,Ty_n)+b\varepsilon_n$$
$$\leq b^2\alpha[\alpha d(Sx_{n-1},Sy_{n-1})+b\alpha H(Sx_{n-1},Ty_{n-1})+H(Ty_{n-1},Sy_n)]+ b^2\alpha H(Sx_n,Ty_n)+b\varepsilon_n$$
$$\leq b^2\alpha^2 d(Sx_{n-1},Sy_{n-1})+(b^2\alpha^2)bH(Sx_{n-1},Ty_{n-1})+b^2\alpha\,\varepsilon_{n-1}+ b^2\alpha H(Sx_n,Ty_n)+b\varepsilon_n$$
$$\leq (b\alpha)^2[d(Sx_{n-1},Sy_{n-1})+(b\alpha)^2 b\, d(Sx_{n-1},Ty_{n-1})+b^2\alpha H(Sx_n,Ty_n)]+ b(\alpha b\varepsilon_{n-1}+b\varepsilon_n)$$

where $k = bq < 1$

Repeating this process (n-1)-times we get (1)

$$d(Sx_{n+1},Sy_{n+1}) \leq (b\alpha)^{n+1} d(Sx_0,Sy_0)+b^2\alpha\sum_{r=0}^{n}(b\alpha)^{n-r} d(Sx_r,Ty_{r+1})+ b\sum_{r=0}^{n}(b\alpha)^{n-r}\varepsilon_r$$

$$d(z,Sy_{n+1}) \leq bd(z,Sx_{n+1})+bd(Sx_{n+1},Sy_{n+1})$$

$$\leq bd(z,Sx_{n+1})+b(b\alpha)^{n+1}d(x_0,y_0)+b^3\alpha\sum_{r=0}^{n}(b\alpha)^{n-r}d(Sx_r,Tx_r)+$$

$$b^2\sum_{r=0}^{n}(b\alpha)^{n-r}\varepsilon_r$$

Assume $\lim_{n\to\infty} Sy_n = z.$

$$\varepsilon_n = H(Sy_{n+1},Ty_n) \leq b\,d(Sy_{n+1,}Sx_{n+1})+bH(Tx_n,Ty_n)$$

$$\leq bd(Sy_{n+1},Sx_{n+1})+[\alpha d(Sx_n,Sy_n)+b\alpha H(Sx_n,Tx_n)]$$

Letting $n\to\infty$, we obtain $\varepsilon\to 0$

Suppose that $\lim_{n\to\infty}\varepsilon_n = 0.$ since $0\leq b\alpha<1$ and $\alpha b<1$

And $\lim_{n\to\infty} Sx_n = z,$ from (1) we obtain (2) by applying lemma (3.2.5)

$$\lim_{n\to\infty} d(z,Sy_{n+1}) \leq \lim_{n\to\infty}[b^3\alpha\sum_{r=0}^{n}(b\alpha)^{n-r}d(Sx_r,Tx_r)+b^2\sum_{r=0}^{n}(b\alpha)^{n-r}\varepsilon_r$$

Let A denote the lower triangular matrix with entries $a_{nr}=(b\alpha)^{n-r}$, then $\lim_{n\to\infty} a_{nr}=0$

for each r and $\lim_{n\to\infty}\sum_{r=0}^{n}a_{nr} = \lim_{n\to\infty}(\frac{1-(b\alpha)^{n+1}}{1-b\alpha}) = \frac{1}{1-b\alpha}.$ Therefore,

A is multiplicative i.e for any converge sequence

$\{b_n\}, \lim_{n\to\infty} A(b_n) = \frac{1}{1-b\alpha}\lim_{n\to\infty} b_n.$ (of, P.692) B. E. Rhoade 1993. Thus,

$$\lim_{n\to\infty} b^3\alpha\sum_{r=0}^{n}(b\alpha)^{n-r}d(Sx_r,Tx_r)=0.$$

In theorem (3.3.1) when we put $S = Id$ (Identity map)

Corollary (3.2):

Let (X,d) be a b-metric space. Suppose that TX is a complete subspace of X and let $T:X\to CL(X)$ and suppose that T is q-multivalued quasi-contraction. Let z be a fixed point of T i.e. $z\in Tz$. For any $x_0\in X$ let the sequence $\{x_n\}$ generated by $x_{n+1}\in Tx_n$ converge to z. Let $\{y_n\}\subseteq X$ be an arbitrary sequence and define $\varepsilon_n = H(y_{n+1},Ty_n),\ n=0,1,2,...,\ k=bq<1$ and $q<\dfrac{1}{b^2+b}$. Then

1)

$$d(z,y_{n+1})\le bd(z,x_{n+1})+b(b\alpha)^{n+1}d(x_0,y_0)+b^3\alpha\sum_{r=0}^{n}(b\alpha)^{n-r}d(x_r,Tx_r)+$$

$$b^2\sum_{r=0}^{n}(b\alpha)^{n-r}\varepsilon_r$$

2) Further, If Tz is singleton then $\lim_{n\to\infty} y_n = z$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0.$

When S and T are a single-valued maps we obtain the corollary (3.3.3).

Corollary (3.3):

Let (X,d) be b-metric space. Suppose that SX or TX is a complete subspace of X and let $S:X\to X$ and $T:X\to X$ and suppose that the hybrid pairs (T, S) is q-singlevalued quasi-contraction let z be a coincidence point of T and S i.e. $z=Su=Tu$ for any $x_0\in X$ let the sequence $\{Sx_n\}$ generated by $Sx_{n+1}=Tx_n$ converge to z, let $\{Sy_n\}\subseteq X$ be an arbitrary sequence and define $\varepsilon_n = d(Sy_{n+1},Ty_n), n=0,1,2,...,.K=bq<1$ and $q<\dfrac{1}{b^2+b}$.Then:

1) $$d(z,Sy_{n+1})\le bd(z,Sx_{n+1})+b(b\alpha)^{n+1}d(x_0,y_0)+b^3\alpha\sum_{r=0}^{n}(b\alpha)^{n-r}d(Sx_r,Tx_r)+$$

2) If Tz is singleton then $\lim_{n\to\infty} y_n = z$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

If $S = I$ (the identity map) we obtain the corollary (3.4) as follows

Corollary (3.4):

Let (X, d) be b-metric space. Suppose that TX is a complete subspace of X and let $T:X \to X$ and suppose T is q-singlevalued quasi-contraction let z be a fixed point of T i.e. $z = Tz$ for any $x_0 \in X$ let the sequence $\{x_n\}$ generated by $x_{n+1} = Tx_n$ converge to z, let $\{y_n\} \subseteq X$ be an arbitrary sequence and define $\varepsilon_n = d(y_{n+1}, Ty_n), n = 0,1,2,...,$.$K = bq < 1$ and $q < \frac{1}{b^2 + b}$ .Then

1)

$$d(z, y_{n+1}) \le bd(z, x_{n+1}) + b(b\alpha)^{n+1} d(x_0, y_0) + b^3\alpha \sum_{r=0}^{n} (b\alpha)^{n-r} d(x_r, Tx_r) + b^2 \sum_{r=0}^{n} (b\alpha)^{n-r} \varepsilon_r$$

$$b^2 \sum_{r=0}^{n} (b\alpha)^{n-r} \varepsilon_r$$

2) If Tz is singleton then $\lim_{n\to\infty} y_n = z$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0.$

Remark (3.5)

1) Theorem (3.1) generalized and extension of theorem (3.2)Hashim (2011), and theorem of S.L singh et.al. (1995). It is also a generalization and extension of theorem (4.3) of Singh et.al(2008). Indeed theorem (3.3.1) is a generalization and extension of a multivalued of results in the literature pertaining to single-valued and multivalued cases .

2) Corollary (3.4) with $b = 1$ generalized Ostrowski's stability theorem (1967) single-valued map.

Reference

Czerwik, S. ,Dlutek, K. and Singh, S. L. (2002)Round-off stability of iteration procedures for set-valued operators in b-metric spaces, J. Natur. Phys. Sci.15 1-8.

Czerwik, S. ,Dlutek,K. and singh,S. L., (1997) Round-off stability of iteration procedures for operators in b-metric spaces, J. Natur. phys. Sci. 11, 87-94.

Czerwik,S. , (1998) Nonlinear set-valued contraction mappings in b-metric space, Attisem.Mat. Fis. Univ. Modena 45(2) 263-276.

Harder,A. M. , and Hicks, T. L., (1988 b) Stability results for fixed point iteration procedures, Math. Japon.33, 693-706.

Harder,A. M. , and Hicks, T.L., (1988 a) A Stable iteration procedure for non-expansive mappings, Math. Japon.33, 687-692.

Hashim, A. M., (2011) Stability of iteration procedures for hybrid maps in b-metric space. Basra Journal of science A. 29(1), 74-84.

Ostrowski,A. M., (1967) The round off stability of iterations, Z. Angew.Math. Mech. 47, 77-81.

Rao.K. P. R. and Swamy P. R.,(2013) A coincidence point theorem for two hybrid pairs with quasi-contraction in b-metric space.Vol. 3(1), Jan-2013.

Rhoades, B.E., and Saliga, L., (2001) Some fixed point iteration procedures II, Nonlinear Anal. From 6 no.1, 193-217.

Roahdes, B. E., (1990) Fixed point theorems and stability results for fixed point iteration procedures, Indian J. pure Appl. Math. 21, 1-9.

Roahdes, B.E., (1993) Fixed point theorems and stability results for fixed point iteration procedures-II, Indian J. pure Appl. Math. 24, 697-703

Singh, S.L. and Prassad B., (2008)Some coincidence theorem and stability of iteration procedures, computers and mathematics with Applications 55 2512-2520.

Singh, S.L. and Chadha, V., (1995) Round off stability of iterations for multivalued operators, C. R. Math.Rep.A cad. Sci. Canada 17(5), 187-192.

Urab,M., (1956)Convergence of numerical iteration of equations, J. Sci. Hiroshima Univ. Ser. A, 19, 479-489.

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الخلاصة

تضمن هذا البحث برهان استقراريه الدوال الهجينة من النوع **Quasi-Contraction** في الفضاء المتري ب وكذلك تم تعميم وتطوير نتائج العديد من الباحثين أمثال

Czerwik*et.al.*1997, Singh*et.al.* 2005,Singh *et.al.* 2008 and Hashim 2011.