# The Maximal Number of Vertices of The Reflexive Polytope

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## Abstract
In this work the basic concept of reflexive polytopeand Gorenstein polytope and Fano with some theorems about them are given. Three standard binary operations on polytopes which are: the cartesian product $(\times)$, the direct sum $(\bigoplus)$ and the free join$(\bowtie)$ are examined. Finally techniques are used to prove the conjecture $|V(P)| \leq 6^{\frac{d}{2}}$ of the maximal number of vertices for a simplicial reflexive polytopes of even dimension.

## اكبر عدد من الرؤوس لمتعدد السطوح الانعكاسي

**الخلاصة**
في هذا البحث عرفنا بعض المفاهيم الاساسية لمتعدد السطوح الانعكاسي و متعدد السطوح جورنستين ومتعدد السطوح فانو, وثلاث من العمليات الابتدائية على متعددات السطوح وهي الضرب الديكارتي $(\times)$, و free sum $(\bigoplus)$ , و free join $(\bowtie)$ تم اختبارها. واخيرا استخدمنا تكنيك لأثبات المسألة المفتوحة $|V(P)| \leq 6^{\frac{d}{2}}$لأكبر عدد من الرؤوس لمتعدد السطوح الانعكاسي.

## INTRODUCTION
The reflexive polytopes $P \subseteq R^d$ were introduced by Victor Batyrev in the context of a mirror symmetry fascinating phenomenon in string theory. These polytopes are special case of an integral polytopes which have been intensively studied and classified by mathematician and physicists alike. By now, all isomorphism classes of reflexive polytope in dimension four, nearly half a billion, are known! Despite all these efforts, still many questions remain open **[1]**. There are a lot of applications concerned with subjects of polytope, for more details see **[2]**, **[3]** and **[4].**Theory of linear inequalities is closely related to the study of convex polytopes, if the bounded subset P of Euclidean d-space has non-empty interior and is determined by i linear inequalities in d variables, then P is a d-dimensional convex polytope (here called a d-polytope) which may have as many as faces of dimension d-1 and the vertices of this polytope are exactly the basic solution of the system of inequalities. So the number of vertices of a convex polytope defined by a system of linear inequalities is crucial for bounding the run-time of exact generation methods. It is not

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easy to achieve a good estimator, since this problem belongs to the complexity class. Thus, to obtain an upper estimate of the size of the computation problem which must be faced in solving the system of linear inequalities, it suffices to find an upper bound for the number of vertices,**[5]**.

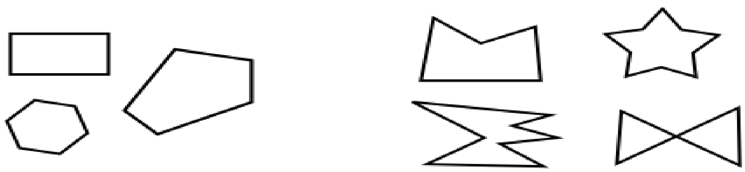
## BASIC CONCEPTS

I n this section some basic definitions of reflexive polytopeand Gorenstein polytope and Fano with some theorems are given.

**Definition(1.1),[6]:**

A set $S \subseteq R^d$ is said to be convex if the entire line segment between any two vectors in S is contained in S. so S is a convex if and only if

$\{\lambda x_1 + (1-\lambda)x_2 : 0 \le \lambda \le 1\} \subseteq S$ for every $x_1, x_2 \in S$.



**Convex Non-convex**

**Figure (1): Convex and non-convex polytpe**

**Definition (1.2),[7]:**

The set of solutions

$$P=\left\{ (x_1,\dots x_d) \in R^d \;\middle|\; \begin{array}{r} a_{11}x_1+\cdots+a_{1n}x_d \le b_1 \\ . \\ . \\ . \\ a_{m1}x_1+\cdots+a_{md}x_d \le b_m \end{array} \right\}$$

to the system

$$\begin{array}{r} a_{11}x_1+\cdots+a_{1n}x_d \le b_1 \\ . \\ . \\ . \\ a_{m1}x_1+\cdots+a_{md}x_d \le b_m \end{array}$$

of a finitely many linear inequalities here ($a_{ij}$ and $b_j$ are real numbers) are called a polyhedron

**Definition (1.3),[8]:**

A polyhedron $P \subseteq R^d$ is bounded if exist $\omega \in R_+^d$ such that $\|x\| \le \omega$ where $x \in P$ then P is polytope.

**Definition (1.4),[1]:**

The polytope $P \subseteq R^d$ is called integral if all vertices of P belong to $Z^d$.

**Definition (1.5),[9]:**

Let P be a polytope in $R^d$ containing the origin in the interior int(P) , convex set $P^*$ in $R^d$ is defined by $P^* \coloneqq \{u \in R^d : \langle u, x\rangle \le 1 \forall x \in P\}$this polytope is called dual of P and $P^{**} = P$.

**Definition(1.6),[1]:**

A polytope P is called a reflexive, if there exists $w \in \mathrm{int}(P \cap Z)$such that all facets have lattice distance 1 from w.

**Definition (1.7),[10]:**

A polytope $P \subseteq R^d$ is said to be simple if there are exactly d edges through each vertex, and it is called simplicial if each facet contains exactly d vertices.

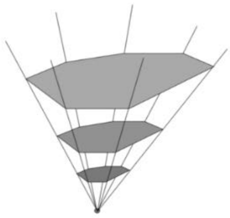
**Definition (1.8),[1]:**

A subset $C \subseteq R^d$ is a cone if for all x , $y \in C$ and $\lambda, \mu \in R_{\ge}$ also $\lambda x + \mu y \in C$. A cone C is polyhedral (finitely constrained) if there are $\propto_1, \dots, \propto_m \in (R^d)^*$ such that

$$C = \bigcap_{i=1}^{m} H^-_{\propto_i} = \{x \in R^d \mid \propto_i (x) \le 0 \text{ for } 1 \le i \le m\}$$

A cone C is called finitely generated by vectors $v_1, \dots, v_r \in R^d$ if

$$C = \mathrm{cone}(v_1, \dots, v_d) \coloneqq \{\sum_{i=1}^{d} \lambda_i v_i | \lambda_i \ge 0 \text{ for } 1 \le i \le d\}$$



**Figure(2): Cone**

**Definition (1.9), [11]:**

A pointed cone $K \subseteq R^d$ is a set of the form K = {v+$aw_1 + bw_2 + \cdots + mw_n$ : $a, b, \dots, m \ge 0$ }, where v,$w_1, w_2, \dots, w_m \in R^d$ such that there exist a hyperplane H for which, $H \cap K = \{v\}$, the cone is called rational if v,$w_1, w_2, \dots, w_m \in Q^d$.

**Definition (1.10),[1]:**

Let $C \subseteq R^{d+1}$ be a (d+1)-dimensional pointed cone. Then C is called Gorenstein cone$C_P$, if there is a d-dimensional integral polytope $P \subseteq R^d$ such that $C \cong C_p$

**Definition (1.11),[9]:**

A polytope is called centrally symmetric if $v \in P$ implies $-v \in P$ .

**Definition(1.12),[12]:**

Let $P \subseteq R^d$ be integral d-polytope. For $t \in Z^+$, the set $tP = \{tX : X \in P\}$ is said to be a dilated polytope.

**Theorem (1.1)(Ehrhart'stheorem),[11]:**

If P is an integral convex d-polytope then L(P,t) is a polynomial in t of degree d

**Definition(1.13),[13]:**

Let $P \subseteq R^d$ be an integral d-polytope. Define a map $L(P,.) : N \to N$ by $L(P,t) = \#(tP\cap Z^d)$, wher"# or card" means the cardinality of ( $tP\cap Z^d$) and N is the set of natural numbers. It is seen that L(P,t) can be represented as: $L(P,t) = \sum_{i=0}^{d} c_i t^i$, with coefficients $c_i$ this polynomial is said to be Ehrhart polynomial of integral d-polytope.

**Definition(1.14),[14]:**

Let $P \subseteq R^d$ , a d-dimension integral polytope, and set $L(P,t) = \#(P \cap Z^d)$ for $t \in Z_{\geq 1}$ .
The Ehrhart series of P is,
$Ehrp(x)=1+\sum_{t\in Z\geq 1} L(P,t)x^t$

**Theorem (1.2),[1]:**

The following are equivalent for a d-dimensional integral polytope$P \subseteq R^d$ of degree s and codgree r :
(1)$C_p^{\Delta}$ Gorenstein cone
(2) rP is reflexive
(3)$\forall k \geq r: \mathrm{int}(kP) \cap Z^d = w + (t-r)P \cap Z^d$ for some $w \in (rP) \cap Z^d$
(4)$L(P,-k) = (-1)^d L(P^{\circ}, k-r) \forall k \in Z$
(5) $Ehr_p(t) = (-1)^{d+1} Ehr_p(t)$
(6) $h_i = h_{s-i}\ \forall i = 0, \dots, s$

**Definition (1.15),[1]:**

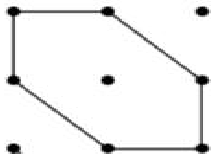
P is Gorenstein polytope if (1)-(6) hold. In other word, an integral polytope P is Gorenstein polytope if some multiple tP is reflexive.
For example, reflexive polytopes are precisely Gorenstein polytopes of degree one.

P 2P

**Figure(3): Gorenstein polytope of codegree r codeg(P) = 2**

**Definition(1.16),[15]:**

An integral convex polytope $P \subseteq R^d$ of dimension d is a Fanopolytope if the origin of $R^d$ is a unique point belonging to the interior of P .



**Figure(4): Fanopolytope**

**Standard Constructions**

Our main motivation is to determine the maximal number of vertices of a simplicial reflexive polytope, and to find out how the exetermal polytopes look like. For this, present a direct way how to construct higher-dimensional reflexive polytopes, which are the cartesian product $(\times)$ , free-sum construction $(\oplus)$ , and the free join $(\bowtie)$.

Product of $P_1$ and $P_2$ to be polytope

$$P_1 \times P_2 = \mathrm{conv}(\{(x, y) \in R^{d+q} | x \in P_1, y \in P_2\})$$

if $0 \in$ relint$(P_1)$ and $0 \in$ relint$(P_2)$, where relint(P) means relative interior which is relint (P) = $\{\sum_{i=1}^{d} \lambda_i v_i | \text{each } \lambda_i > 0, \sum_{i=0}^{d} \lambda_i = 1\}$,

Then the polytope

$$P_1 \oplus P_2 = \mathrm{conv}(\{(x, 0_d) \in R^{d+q} | x \in P_1\} \cup \{(0_q, y) \in R^{d+q} | y \in P_2\})$$

Defines the direct sum (or free sum) of $P_1$and$P_2$.

Finally let $P_1 \subseteq R^{d`+q+1}$ be a d-polytope and $P_2 \subseteq R^{d+q+1}$ be q-polytope if aff$(P_1)$ and aff$(P_2)$ are skew that they do not intersect and contain no parallel lines then the free join of the polytopes is

$$P_1 \bowtie P_2 = \mathrm{conv}(P_1 \cup P_2)$$

Some simple visual examples of these operations are useful for given which are.

**Example (2.1),[16]:**

Let $P_1 = \mathrm{conv}((1), (3)) \subseteq R$, be a line segment of length 2, and let $P_2 = \mathrm{conv}((-1, -1), (1, -1), (1,1)(-1,1)) \subseteq R^2$, be a two-by-two square. The Cartesian product are constructing

$$P_1 \times P_2 = \mathrm{conv}\begin{pmatrix}(1, -1, -1), (1,1, -1), (1,1,1), (1, -1,1) \\ (3, -1, -1), (3,1, -1), (3,1,1), (3, -1,1)\end{pmatrix}$$

is a cube in $R^3$.

**Figure(5): Cartesian product**

**Example (2.2),[16]:**

Let $P_2$ be define as the above example, but $P_1 = \mathrm{conv}((-1),(1)) \subset R$ so that the origin is in the relative interior of both polytopes, we can now construct the dirct sum

$$P_1 \oplus P_2 = \mathrm{conv}(\{(0,-1,-1),(0,1,-1),(0,-1,1)\} \cup \{(-1,0,0),(1,0,0))$$

is a bipyramid in $R^3$.

$P_1$ and $P_2$ $\qquad$ $P_1 \oplus P_2$

**Figure(6): The free sum**

**Example (2.3),[16]:**

Unfortunately, the free join of a square and a line is four-dimensional, so we will have to simplify the setup of the previous two examples a bit to be able to visualize it. Let

$P_1$ =conv((0,-1,1),(0,1,1)) and let $P_2$ =conv((-1,0,0),(1,0,0,)) then

$$P_1 \bowtie P_2 = \mathrm{conv}(\{(0,-1,1),(0,1,1)\} \cup \{(-1,0,0),(1,0,0)\})$$

Is simplex in $R^3$.

$P_1$ and $P_2$ $\qquad$ $P_1 \bowtie P_2$

**Figure(7): Free join**

It follows directly from the definition that the free-sum construction is the dual operation to products:

$$(P_1 \oplus P_2)^* = P_1^* \times P_2^*$$

In particular if $P_1$ and $P_2$ are reflexive, and then $P_1 \oplus P_2, P_1 \times P_2$ are reflexive.

There are nice formulas how the Ehrhart and h-polynomials behave under the free sum and product construction. This shows the relation between the free sum and Cartesian product in the following proposition.

**Proposition (2.1),[1]:**

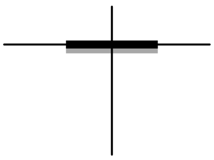
Let$P_1$ and$P_2$ be a reflexive polytope. Then

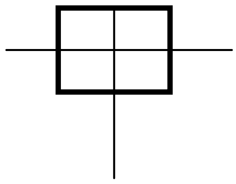
$$L(P_1 \times P_2, t) = L(P_1, t) \cdot L(P_2, t),$$

$h_{P_1 \oplus P_2} = h_{P_1} h_{P_2}.$

**Example(2.4):**

Let $P_1 = \operatorname{conv}\big((1), (-1)\big) \subseteq R$, be a line segment of length 2, and $P_2 = \operatorname{conv}((-1,-1), (1,-1), (1,1)(-1,1)) \subseteq R^2$





$\mathbf{P_1}$ $\mathbf{P_2}$

**Figure(8): Two reflexive polytopes**

$L(P_1, t) = 2t + 1$ , $L(P_2, t) = 4t^2 + 4t + 1$.Then $L(P_1, t) \cdot L(P_2, t) = 8t^3 + 12t^2 + 6t + 1$.

$P_1 \times P_2 = \operatorname{conv}\begin{pmatrix} (1,-1,-1), (1,1,-1), (1,1,1), (1,-1,1) \\ (-1,-1,-1), (-1,1,-1), (-1,1,1), (-1,-1,1) \end{pmatrix}$ as cube in 3-dimension.

$L(P_1 \times P_2) = 8t^3 + 12t^2 + 6t + 1.$

$h_{P_1} = t + 1$ , $h_{P_2} = t^2 + 5t + 1$. Then $h_{P_1} h_{P_2} = 3t^3 + 6t^2 + 6t + 1$

$P_1 \oplus P_2 = \operatorname{conv}\big((0,1,1), (0,1,-1), (0,-1,1), (0,-1,-1) \cup (1,0,0), (-1,0,0)\big) \subseteq R^3$.

Then $h_{P_1 \oplus P_2} = 3t^3 + 6t^2 + 6t + 1$

**The combinatory of simplicial reflexive polytopes**

Let $P \subseteq R^d$ be a d-dimensional reflexive polytope with the origin in its interior. Now, let us turn to a simplicial reflexive d-polytopes. Due to the complete classification of reflexive d-polytopes $d \geq 4$, the simplicial ones have been classified as well. In dimension two there are (of course) 16 isomorphism classes of simplicial reflexive polytopes, while in dimension three and four there are 194 and 5450 isomorphism classes respectively. Reflexivity guarantees that $u_F \in Z^d$ for every facet F of a simplicial reflexive polytope P. But in general, the points $u_F^w \notin Z^d$ for arbitrary facet F of P and $w \in V(F)$. Where V(F) is the set of vertices of the face for the polytope , However,

$\{u_F^w \mid w \in V(F)\} \subset Z^d \Leftrightarrow V(F)$ is a basis of $Z^d$.

The fact that for any $x \in V(P)$,

$\langle u_F, x \rangle \leq 0$ if and only if $x \notin F$ .

**Note (3.1),[9]:**

When P is any polytope, $v_P$ is defined to be the sum of the vertices of P,

$$v_P := \sum_{v \in V(P)} v\,.$$

**Definition (3.1),[9]:**
Let P be a polytope containing the origin in its interior. A facet F of P is called special, if $v_P$ is a non-negative linear combination of V(F), where V(F) means the vertices of the face F.
Here are some obvious properties of special facets of polytopes having the origin in its interior.

**Lemma (3.1),[9]:**
Let P be a d-polytope with $0 \in \text{int}(P)$, then
1. P has at least one special facet.
2. Every facet of P is special if and only if $v_P = 0$
3. If P is simplicial, then a facet F is special if and only if $\langle u_F^v, v_P \rangle \geq 0$ for all v $\in V(F)$
4. If P is simplicial and reflexive, then $0 \leq \langle u_F, v_P \rangle \leq d-1$ for any special facet F of P.

**Bounds on the number of vertices [9]:**
When P is an arbitrary reflexive d-polytope the following bound on the number of vertices:$|V(P)| \leq 2d\alpha$, where $\alpha = \max\{|V(F)| : \text{F facet of P}\}$. It is conjectured that $|V(P)| \leq 6^{\frac{d}{2}}$ for any reflexive d-polytope with equality if and only if d is even and $P^*$ is isomorphic to the convex hull of the points
$\pm e_1 \pm e_2, \dots, \pm e_d, \pm(e_1 - e_2), \dots \pm(e_{d-1} - e_d)$.
Where $e_i$ are Z-basis.
For simplicial reflexive polytope a theorem is given.

**Remark(3.1.1)**
If $u \in V(P^*)$, we denoted by $F_u$ the corresponding facet of P, namely

$$F_u = \{x \in P | \langle x, u \rangle = -1\}.$$

$\delta_P$ is defined as

$$\delta_P := \min\{\langle v, u \rangle | v \in V(P), u \in V(P^*)\} \in Z_{\geq 0}$$

**Lemma (3.1.1), [17]:**
Let P be a simplicial reflexive polytope and $v \in V(P)$, $u \in V(P)^*$ such that $\langle v, u \rangle = \delta_P$. Then v is adjacent to $F_u$

**Theorem(3.1.1),[17]:**
Let P be simplicial reflexive polytope of dimension d then:

$$|V(p)| \leq 3d \qquad \dots(1)$$

With equality hold if and only if d is even and$X_p \cong (S_3)^{\frac{d}{2}}$, where $X_P$means Gorenstein toric Fano variety, and $S_3$ is symmetric group.

**Proof**:
First of all, observe that for any $u \in V(P^*)$ we have

$$|\{v \in V(P) | \langle v, u \rangle = -1\}| = d \qquad \dots(2)$$

and

$|\{v \in V(P)|\langle v,u\rangle = 0\}| \leq d$ ….(3)

In fact, since P is simplicial, the facet $F_u$ contains n vertices. Moreover, if $\langle v,u\rangle = 0$, then $\delta_p = 0$,
by lemma(3.1.1)it is know that v is adjacent to $F_u$. Again, since P is simplicial, $F_u$ has at most n adjacent vertices, and the equality (1) .
The origin lies in the interior of $P^*$, the relation can be written as,

$m_1u_1 + \cdots + m_hu_h = 0$ ...(4)
Where
$h > 0$, $u_1,\dots,u_h$ are vertices of $P^*$, and $m_1,\dots,m_h$ arepositive integer.
Set I:= $\{1,\dots,h\}$ and M :=$\sum_{i\in I} m_i$ , for any vertex v of P, define
A(v) :=$\{i \in I|\langle v,u_i\rangle = -1$and B(v) :=$\{i \in I|\langle v,u_i\rangle = 0\}$.
Then observe that $\langle v,u_i\rangle \geq 1$ for any $i \notin A(v) \cup B(v)$. So for every $v \in V(P)$ we have,

$$0 = \sum_{i\in I} m_i\langle v,u_i\rangle = -\sum_{i\in A(v)} m_i + \sum_{i\notin A(v)\cup B(v)} m_i\langle v,u_i\rangle$$
$$\geq -\sum_{i\in A(v)} m_i + \sum_{i\notin A(v)\cup B(v)} m_i = M - 2\sum_{I\in A(v)} m_i - \sum_{i\in B(v)} m_i$$

Summing over all vertices of P we get,

$$M|V(P)| \leq 2\sum_{v\in V(P)}\sum_{i\in A(v)} m_i + \sum_{v\in V(P)}\sum_{i\in B(v)} m_i$$
$$= 2\sum_{i\in I} m_i|\{v \in V(P)|\langle v,u_i\rangle = -1\} + \sum_{i\in I} m_i\,|\{v \in V(P)|\langle u,u_i\rangle = 0\}|.$$

And using (2) and (3) this gives$|V(p)| \leq 3d$.
Assuming that$|V(P)| = 3d$, Then all inequalities above are equalities; in particular, for any v and $u_i$ such that$\langle v,u_i\rangle > 0$, must have$\langle v,u_i\rangle = 1$.
Observe now that are can choose a relation as (4) involving all vertices of $P^*$, namely with h = $|V(P^*)|$, so $\langle v,u\rangle \in \{-1,0,1\}$ for every $v \in V(P)$ and $u \in V(P^*)$. Then P and $P^*$ are centrally symmetric.

**The maximal number of vertices**

Before we explain our goal we show some definitions needed in our work.

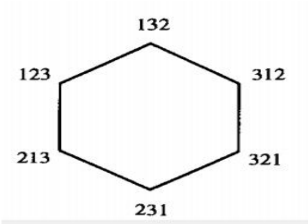
**Definition (3.2.1),[17]:**

If $G \subseteq S_n$ is a permutation group of degree n then its permutation representation $\sigma: G \to GL(n,R)$ is defined as follows. If x is an element in G, then the (i,j)-th entry of σ(x) is given by

$$\sigma(x)_{ij} = \begin{cases} 1 \text{ if } \sigma \text{ sends i to j} \\ 0 \text{ otherwise} \end{cases}$$

If $x \in G \subseteq S_n$ then σ(x) is n×n matrix with n once and $n^2 - n$ zeros. Furthermore, each row and each column contains exactly one nonzero entry.

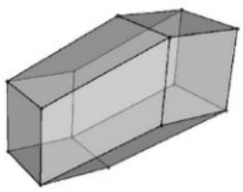
Now, the symmetric group can be represented as the face of permutahedron in two-dimension (defined as the convex hull of all vectors that are obtained by permuting the coordinates of the vector $\begin{pmatrix}1\\ \vdots\\ d\end{pmatrix}$). This is given in figure (9).




**Figure(9): A hexagon, face of permutahedron**

It is a natural question to ask for the maximal number of vertices of a d dimensional reflexive (or Gorenstein) polytope. Let us look at small dimension d$\leq$ 4 where the answer is known by the classification of Kreuzer and skarke.

**Example (3.2.3):**
For d=2 :$|V(p)| \leq 6$, only attained by the reflexive hexagon $\mathcal{H}$.
For d=3 :$|V(p)| \leq 14$, only attained by the polytope in figure (10).
For d=4 :$|V(p)| \leq 36$, only attained by $\mathcal{H} \times \mathcal{H}$.
Based upon these observations we state the following daring conjecture.



**Figure(10): polytope in 3-dimension**

**Theorem(3.2.1),[1]:**
Let P be a simplicial. Then $|V(P)| \leq 3d$, and equality hold only if d is even and P $\cong \mathcal{H} \circ \ldots \circ \mathcal{H}$.

**Proof**:
let F be a special facet. Obviously, special facets exist. Let us slice the polytope (for i$\in \{-1,0,1,\ldots\}$) :
$H_p(F,i) := \{v \in V(p) \langle \eta_F \rangle v = i\} \forall i \in Z_{\geq -1}$
Clearly,
$\left|H_p(F,0)\right| = d.$
Moreover
$|H_p(F,1) \leq d.$
By definition of special facet we have the following inequality:

$$0 \geq \langle \eta_F, \sum_{v \in V(p)} v \rangle = \sum_{i \geq -1} i\left|H_p(F,i)\right| = -d + \sum_{i \geq 1} i\left|H_p(F,i)\right|.$$

Hence, simply count the vertices as:

$$|V(P)| = \sum_{I \geq -1} |H_P(F,i)| \leq |H_p(F,-1) + |H_p(F,0) + \sum_{I \geq 1} |H_p(F,i) \leq 3d\,.$$

It remains to consider the equality case |V(p)| = 3d.
In this case, inequality in equation (1) yields that

$$\langle \eta_F, \sum_{v \in V(p)} v \rangle = 0$$

Hence$\sum_{v \in V(p)} v = 0$, so any facet of P is special. Moreover, equalities in equation (1) and show that any vertex of P lies in $H_p$(F,i) for i = -1,0,1. Therefore, $-\eta_F \in P^*$.so $-P^* \subseteq P^*$ and thus$-P^* = P^*$. In other word, P is centrally symmetric.
Since |V(P)|= 3d and $|H_p$(F,1)| = $|H_p$(F,-1)| = d, we have $|H_p$(F,0)| =d. then$H_p$(F,0) =$\{m_1, \dots m_2\}$,
as defined in the previous subsection. Let K$\in \{1, \dots k\}$, by proposition there are I, J $\subseteq$ {1,…,d}, $I \cap J = \emptyset$, $|I| = |J|$ such that $m_k = \sum_{j \in J} b_j - \sum_{i \in I} b_i$. Since all $m_1, \dots, m_d$ are pairwise different, lemma yields that $m_k = b_{jk} - b_k$ for some$j_k \in \{1, \dots, d\}$, $j_k \neq k$ in other words,

$$\sigma : \{1, \dots, d\} \to \{1, \dots, d\} K \to j_k$$

is a fixed-point free($\sigma(i) \neq i$) n volution ($\sigma^2 = 1, j_k = K$). It may assume that this permutation is of the form

$$\sigma = (12)(34) \dots (d-1d).$$

In particular, d is even. Moreover
P = conv($\pm b_1$,$\pm(b_1$ - $b_2)$,$\pm b_2$, . . . ,$\pm b_d$ ,$\pm(b_{d-1} - b_d$ ),$\pm b_d$ ).
It remains to show that $b_1, \dots, b_d$ is a integral basis, since it that caseP$\cong \mathcal{H}_{\circ} \dots {}_{\circ}\mathcal{H}$.
Now, according to conjecture [1 ] the open problem about the maximal number of vertices is given with

**Theorem (3.2.2):**

Let $P \subseteq R^d$ be asimplicial reflexive polytope V(P) it vertices, and the hexagon $\mathcal{H}$ is the face for the reflexive polytope then

$|V(p)| \leq 6^{\frac{d}{2}}$,

Equality holds for d even and P$\cong \mathcal{H}^{\frac{d}{2}}$.

**Proof:**

Since P is simplical, then using theorem (3.2.1), we get
$|V(P)| \leq 3d$, where d is even, and $P \cong \mathcal{H} \circ \mathcal{H} \circ \cdots \circ \mathcal{H}$.

And by theorem (3.1.2) |V(P)| = 3d , if d is even and $X_P \cong S_3^{\frac{d}{2}}$.
By the above two theorems we get:

$|V(P)| \leq 3d$ if d even and $X_P \cong \mathcal{H}^{\frac{d}{2}}$, where $\mathcal{H}$ is isomorphic to $S_3$, that means $\mathcal{H} \cong S_3$.
Also, one can prove that $S_3 \cong \mathcal{H}$ by using the information in section (3.2).

Finally, to prove that $|V(P)| \leq 6^{\frac{d}{2}}$ we use

$|V(P)| \leq 3d \leq 6^{\frac{d}{2}}$ and the induction method when d is even

Let d = 2n then

$3(2n) \leq 6^{\frac{2n}{2}}$

$6n = 6^{n}$.

Let n = r, then

$6r \leq 6^{r}$ is true.

By induction method, we get the proof for this conjecture.

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