Controllability of NonlinearBoundary Value Control Systems in Banach Spaces Using Schauder Fixed Point Theorem

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ABSTRACT

In this paper, sufficient conditions for controllability of nonlinear boundary value control system in Banach spaces are established. The results are obtained by using semigroup theory "compact semigroup"and some techniques of nonlinear functional analysis, such as, Schauder fixed point theorem. Moreover, example is provided to illustrate the theory.

Keywords: Controllability, Banach space, Semigroup theory, Schauder fixed point theorem.

قابلية السيطرة لأنظمة غير خطية ذات شرط حدودي في فضاءات بناخ باستخدام نظرية النقطة الصامدة لشويدر

الخلاصة

في هذا البحث ، تم إثبات المبرهنة التي تتعامل مع الشروط الكافية للقابلية على السيطرة لنظام غير خطي ذات شرط حدودي في فضاءات بناخ وذلك باستخدام نظرية شبه الزمرة(شبه الزمرة المتراصة) وبعض الطرائق التقنية ضمن التحليل الدالي غير الخطي مثل نظرية النقطة الصامدة لشويدر، كذلك تم اعطاء مثال يوضح قيمة النظرية أعلاه .

INTRODUCTION

The theory of semigroup of linear operators lends a convenient setting and offers many advantages for applications. Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after well-developed semigroup theory was at hand. Many scientific and engineering problems can be modeled by partial differential equations, integral equations, or coupled ordinary and partial differential

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equations that can be described as differential equations in infinite–dimensional spaces using semigroups. Nonlinear equations, with and without delays, serve as an abstract formulation for many partial equations which arise in problems connected with heat flow in materials with memory, viscoelasticity, and other physical phenomena. So, the study of controllability results for such systems in infinite–dimensional spaces is important. For the motivation of abstract system and controllability of linear system, one can refer to the [1].

Now, let E and U be a pair of real Banach spaces with norms $\|.\|$ and $|.|$, respectively. Let A be a linear closed and densely defined operator with $D(A)\subseteq E$, $\|A\|\leq C_1$, C_1 is a positive constant , and let x be a linear operator with $D(x)\subseteq E$ and $R(x) \subseteq X$, a Banach space.

In this paper we discuss the controllability of mild solution of the following nonlinear boundary value control problem in arbitrary Banach spaces.

$$\left.\begin{aligned} \dot{x}(t) = Ax(t) + (Bu)(t) + f(t, x(t)) + Q(t, K(t, x(t))),\ t \in J = [0,b] \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0 \end{aligned}\right\} \qquad (1)$$

where

$B_1 : U \to X$ is a linear continuous operator, the control function $u \in L^1(J, U)$, a Banach space of admissible control functions. The nonlinear operators,

$$f \in C(J \times E, E),\ Q \in C(J \times E, E),\ \text{ and } K \in C(J \times E, E)$$

are all uniformly bounded continuous operators. Where the state $x(.)$ takes values in the Banach space X and the control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U a Banach space. Here, the linear operator A generates a compact semigroup $T(t),\ t > 0$, on a Banach space X with norm $\|.\|$, and B is a bounded linear operator from U into X.

Controllability of the above system with different conditions has been studied by several authors. Balakrishnan in [2] showed that the solution of a parabolic boundary control equation with L^2 controls can be expressed as a mild solution to an operator equation. Fattorini in [3] discussed the general theory of boundary control systems. Han and Park in [4] derived a set of sufficient conditions for the boundary controllability of a semilinear system with a nonlocal condition. Al-Moosawy in [5] discussed the controllability of the mild solution for the problem (1) by using Banach fixed point theorem, where $f \equiv 0$, A generates a strongly continuous semigroup (C_0-semigroup) on X and the operators K, Q are satisfying Lipschitz condition on the second argument. Al-Jawari in [6] extended the work in [5] by studying the controllability in quasi-Banach spaces of kind L^p, $0 < p < 1$, using a quasi-Banach contraction principle theorem. From all the above we find a reasonable justification to accomplish the study of this paper. The aim of this paper is to study the controllability of nonlinear problem (1) in Banach spaces by using the Schauder fixed point theorem.

Definitions and Theorems

Before proceeding to main result, we shall set in this section some definitions and theorems that will be used in our subsequent discussion.

Definition 2.1 [7]: A family $T(t), 0 \le t < \infty$ of bounded linear operators on a Banach space X is called a (one-parameter) semigroup on X if it satisfies the following conditions:

$T(t+s) = T(t)T(s), \forall t,s \ge 0$ *and* $T(0) = I$ (I is the identity operator on X).

Definition 2.2[7]: The infinitesimal generator A of the semigroup $T(t)$ on Banach space X is defined by $Ax = \lim_{t\to 0^+}(1/t)(T(t)x - x)$,where the limit exists and the domain of A is

$D(A) = \{x \in X: \lim_{t\to 0^+}(1/t)(T(t)x - x) \ exists\}$.

Definition 2.3[7]: A semigroup $T(t)$, $0 \le t < \infty$ of bounded linear operators on Banach space X is said to be strongly continuous semigroup (or C_0-semigroup) if,

$\|T(t)x - x\|_X \to 0 \ as \ t \to 0^+$ for all $x \in X$.

Definition 2.4 [2]: A semigroup $T(t)$, $0 \le t \le \infty$ is called compact if $T(t)$ is a compact operator for each $t > 0$.

Definition 2.5 [8]: Let X be a Banach space, a subset E of X is said to be totally bounded (or precompact) iff for every $\epsilon > 0$, E may be covered by a finite collection of open balls of radius ϵ.

Definition 2.6 [8]: A subset E of C $[a, b]$ is said to be equicontinuous, if for each $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ, such that for all $f \in E$ and all $x, y \in [a,b]$ satisfying $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Note that δ does not depend on f.

Lemma 2.1 [8]: Let X and Y be normed spaces. Then;

(a) Every compact linear operator $T: X \to Y$ is bounded, hence continuous.

(b) If dim $X = \infty$, the identity operator $I: X \to X$ (which is continuous) is not compact.

Remark 2.1 [7]: A semigroup $T(t)$, $0 \le t < \infty$ on X is called continuous in the uniform operator topology if:-

1. $\|T(t+\delta)x - T(t)x\|_{L(X)} \to 0$, as $\delta \to 0, \forall x \in X$.
2. $\|T(t)x - T(t-\delta)x\|_{L(X)} \to 0$, as $\delta \to 0, \forall x \in X$.

Theorem 2.1 [9]. (Banach Theorem): Every contraction mapping of a Banach space into itself has a unique fixed point.

Theorem 2.2 [9]. (Schauder Theorem): Every continuous operator that maps a closed convex subset of a Banach space into a compact subset of itself has at least one fixed point.

Theorem 2.3 [10]. (**Arzela-Ascoli's Theorem)** Suppose X is a Banach space and E is compact metric space. In order that a subset M of the Banach space $C(E, X)$ be relatively compact, iff M be equicontinuous and that for $x \in E$, the set $M(x) = \{f(x) = f \in M\}$ be relatively compact in X.

Controllability of Nonlinear System (1)

In this section the controllability of the mild solution to the problem (1) in Banach spaces will be studied by using semigroup theory "compact semigroup", and Schauder fixed point theorem.

Preliminaries

Let $\sigma : E \longrightarrow E$ be the linear operator, defined by,

$\sigma x = Ax, \forall x \in D(\sigma)$, where $D(\sigma) = \{x \in D(A) \; ; \; \tau(x) = 0\}$

Now, we shall make the following hypotheses:

1. $D(A) \subset D(\tau)$ and the restriction of X to $D(A)$ is continuous relative to graph norm of $D(\sigma)$.
2. The operator $\boldsymbol{\sigma}$ is the infinitesimal generator of a C_0 compact semigroup $T(.)$ such that $max_{t>0}\|T(t)\| \leq M$, where $M > 0$ is a constant.
3. There exists a linear continuous operator $B_2 : U \rightarrow E$, such that $AB_2 \in L(U,E)$, $x(B_2u)=B_1u \;\; \forall u \in U$. Also $B_2u(t)$ is continuously differentiable, and $\|B_2u\| \leq L\|B_1u\| \forall u \in U$, where L is a constant.
4. For all $t \in (0, b]$ and $u \in U$, $T(t)B_2u \in D(\boldsymbol{\sigma})$. Moreover, there exists a positive function $v \in L^1(0,b)$ such that $\|\boldsymbol{\sigma}T(t)B_2\| \leq v(t), a.e., t \in (0, b)$.
5. $B: U \rightarrow E$ is a bounded linear operator, $\|B\| \leq C$, where C is a positive constant.
6. The nonlinear operators $f(t, x(t)),\ Q(t, K(t, x(t)))$ for $t \in J = [0,b]$, satisfy; $\|f(t, x(t))\| \leq L_1$, $\left\|Q\left(t, K\left(t, x(t)\right)\right)\right\| \leq L_2$, where $L_1 > 0$ and $L_2 > 0$.

Controllability Results for Problem *(1*).

Let $x(t)$ be the solution of problem (*1*), then the following function can be defined,

$$z(t) = x(t) - B_2u(t),$$

and from (1), condition(3), we get that: $\tau z(t) = \tau x(t) - \tau B_2u(t) = B_1u(t) - B_1u(t) = 0$. Thus $z(t) \in D(v)$.

Hence problem (1) can be written in terms of σ and B_2 as;

$$\dot{x}(t) = A[z(t) + B_2u(t)] + Bu(t) + f(t, x(t)) + Q(t, K(t, x(t))), t \in J = [0, b]$$

$$x(t) = z(t) + B_2u(t),\ x(0) = x_0$$

Since $z(t) \in D(\sigma)$ then $\sigma z(t) = A z(t)$, and;

$$\dot{x}(t) = \sigma z(t) + AB_2u(t) + Bu(t) + f(t, x(t)) + Q(t, K(t, x(t))),$$
$$x(t) = z(t) + B_2 u(t),\quad x(0) = x_0. \tag{2}$$

From condition (3), $B_2u(t)$ is continuously differentiable, if x is continuously differentiable on J, then by definition of the mild solution[7], $z(t) = x(t) - B_2u(t)$, can be defined as a mild solution to Cauchy problem:

$$\dot{z}(t) = \dot{x}(t) - B_2\dot{u}(t).$$

By equation (2), one gets that,

$$\dot{z}(t) = \sigma z(t) + AB_2u(t) + Bu(t) + f(t, x(t)) + Q(t, K(t, x(t))) - B_2\dot{u}(t),$$

$$z(0) = x_0 - B_2u(0). \tag{3}$$

Then by condition (2), $T(t)$, $t > 0$ is the C_0 compact semigroup generated by the linear operator σ, and $z(t)$ is a solution of (3), then the function,
$H(s) = T(t\text{-}s)\, z(s)$ is differentiable for $0 < s < t$ [12], and

$$\frac{d}{ds}H(s) = T(t-s)\frac{d}{ds}z(s) + z(s)\frac{d}{ds}T(t-s).$$

Thus by equation (3) [11],

$$\frac{d}{ds}H(s) = T(t\text{-}s)[\sigma z(t) + AB_2u(t) + Bu(t) + f(t,x(t)) + Q(t,K(t,x(t)))$$
$$-B_2\frac{d}{ds}u(s)] + z(s)[-AT(t-s)]$$
$$= T(t-s)\,\sigma z(t) + T(t-s)AB_2u(t) + T(t-s)Bu(t) +$$
$$T(t-s)\, f(t, x(t)) + T(t-s)\, Q(t, K(t, x(t)))$$
$$-T(t-s)B_2\frac{d}{ds}u(s) - T(t-s)Az(t),$$

and since $\sigma z(t) = Az(t)$, we get that

$$\frac{d}{ds}H(s) = T(t-s)AB_2 u(t) + T(t-s)Bu(t) + T(t-s)\, f(t, x(t)) +$$
$$T(t-s)\, Q(t, K(t, x(t))) - T(t-s)B_2\frac{d}{ds}u(s). \tag{4}$$

On integrating (4) from *0* to *t* yields;

$$H(t)-H(0)=\int_0^t T(t-s)AB_2\, u(s)\, ds+\int_0^t T(t-s)Bu(s)\, ds+\int_0^t T(t-s)\, f(s,x(s))\, ds$$
$$+\int_0^t T(t-s)\, Q(s,K(s,x(s)))\, ds-\int_0^t T(t-s)B_2\frac{d}{ds}u(s)\, ds \qquad (5)$$

From the definition of $H(s) = T(t\text{-}s)\, z(s)$, we have:

$$H(t)=T(t-t)z(t)=T(0)[x(t)-B_2u(t)]=x(t)-B_2u(t)\,, \qquad (6)$$

and

$$H(0)=T(t-s)z(0)=T(t)[x_0-B_2\, u(0)]=T(t)x_0-T(t)B_2u(0) \qquad (7)$$

Now, by integrating the term $\int_0^t T(t-s)B_2\frac{d}{ds}u(s)ds$, in equation (5) by parts, we obtain that:

$$\int_0^t T(t-s)B_2\frac{d}{ds}u(s)\, ds=T(t-s)B_2\, u(s)\big|_0^t+\int_0^t u(s)\,\sigma\, T(t-s)B_2\, ds$$
$$=B_2u(t)-T(t)B_2u(0)+\int_0^t u(s)\,\sigma\, T(t-s)B_2\, ds \qquad (8)$$

substituting $(6),(7)$ and (8) in (5), we get :

$$x(t)=T(t)x_0+\int_0^t T(t-s)AB_2u(s)ds+\int_0^t T(t-s)Bu(s)ds+\int_0^t T(t-s)\, f(s,x(s))\, ds$$
$$+\int_0^t T(t-s)\, Q(s,K(s,x(s)))\, ds-\int_0^t \sigma\, T(t-s)B_2u(s)\, ds \qquad (9)$$

According to the above derivation, the following definition may be imposed as follows:

Definition 3.1. A function $x:[0,b]\rightarrow E$, defined by the equation (8), is called a **mild solution** of problem(1) if x is continuous on $[0,b]$, continuously differentiable on $(0,b)$ and $x(t)\in$ E, for $0 < s < t$.

Definition 3.2. The boundary value control problem (1) is said to be **controllable** on the interval $J= [0,b]$ if for every $x_0 ,x_1 \in E$, there exists a control $u\in L^2(J,U)$, such that the solution $x(.)$ of (1) satisfies $x(b)=x_1$.

Further considering the following additional conditions:

7. The linear operator W from $L^2(J, U)$ into E defined by

$$Wu=\int_0^b [T(b-s)AB_2+T(b-s)B-T(b-s)\sigma B_2]\, u(s)\, ds$$

induces an invertible operator $\tilde{W}$ defined *on* $L^2(J,U)/\ker W$ and there exists a positive constant K_2 such that $\|\tilde{W}^{-1}\|\leq K_2$. The construction of the bounded inverse operator $\tilde{W}^{-1}$ in general Banach space is outlined in[11,12].

8. There exists a constant $K_1 > 0$, such that $\int_0^b v(t)\,dt \le K_1$

9. $r = K_2(bMAB_2 + bMC + K_1)[\|x_1\| + N] + N$,
where $N = M\|x_0\| + bM(L_1 + L_2)$

Main Result

In this subsection we will prove the theorem that deals with the controllabilityof the problem (1).

Theorem *3.1.* If the boundary value control problem (1)

$$\dot{x} = Ax(t) + Bu(t) + f(t, x(t)) + Q(t, K(t, x(t))),\ \ t \in J = [0, b]$$
$$\tau x(t) = B_1 u(t),\ x(0) = x_0,$$

and the conditions (1)-(9) hold , then the system (1) is controllable on J.
Proof: From equation (9) we have

$$x_1 = x(b) = T(b)x_0 + \int_0^b T(b-s)AB_2 u(s)\,ds + \int_0^b T(b-s)Bu(s)\,ds$$
$$+ \int_0^b T(b-s) f(s, x(s))\,ds + \int_0^b T(b-s)\,Q(s, K(s, x(s)))\,ds$$
$$- \int_0^b \sigma\, T(b-s)B_2 u(s)\,ds$$

Now by using condition (7), for an arbitrary function *x* (.) the control may be defined by

$$u(t) = \tilde{W}^{-1}[x_1 - T(b)x_0 - \int_0^b T(b-s)\, f(s, x(s))\,ds$$
$$- \int_0^b T(b-s)\,Q(s, K(s, x(s)))\,ds].$$

Now one will show that when using this control, the operator defined by

$$(\Phi x)(t) = T(t)x_0 + \int_0^t T(t-s)AB_2u(s)\,ds + \int_0^t T(t-s)Bu(s)\,ds$$
$$-\int_0^t \sigma T(t-s)B_2u(s)\,ds + \int_0^t T(t-s)f(s,x(s))\,ds + \int_0^t T(t-s)Q(s,K(s,x(s)))\,ds$$

has a fixed point , this fixed point is then a solution of problem (9) . Clearly, $(\Phi x)(b) = x_1$, which means that the control u steers the nonlinear control problem from the initial state x_0 to x_1 in time b, provided one can obtain a fixed point of the nonlinear operator Φ.
Let $Y = C(J,E)$ and $Y_0 = \{x : x \in Y, x(0) = x_0, \|x(t)\| \le r, \text{ for } t \in J\}$,
where the positive constant r is given by
$r = K_2(bMAB_2 + bMC + K_1)[\|x_1\| + N] + N$,
where $N = M\|x_0\| + bM(L_1 + L_2)$
Then, Y_0 is clearly a bounded, closed, convex subset of Y [8]. Now a mapping Φ: $Y \rightarrow Y_0$ is defined by

$$(\Phi x)(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))\,ds + \int_0^t T(t-s)Q(s,K(s,x(s)))\,ds$$
$$+\int_0^t T(t-\eta)(AB_2 + B - \sigma B_2)\tilde{W}^{-1}[x_1 - T(b)x_0 - \int_0^b T(t-s)f(s,x(s))\,ds$$
$$-\int_0^b T(b-s)Q(s,K(s,x(s)))\,ds](\eta)d\eta$$

By taking the norm of both sides, it can be seen that

$$\|(\Phi x)(t)\| \leq \|T(t)x_0\| + \left\|\int_0^t T(t-s) f(s,x(s))\,ds\right\| + \left\|\int_0^t T(t-s)\,Q(s,K(s,x(s)))\,ds\right\|$$
$$+\left\|\int_0^t T(t-\eta)\,(AB_2 + B - \sigma B_2)\,\tilde{W}^{-1}[x_1 - T(b)x_0 - \int_0^b T(t-s)\,f(s,x(s))\,ds\right.$$
$$\left.-\int_0^b T(b-s)\,Q(s,K(s,x(s)))\,ds]\,(\eta)\,d\eta\right\|$$

$$\|(\Phi x)(t)\| \leq \|T(t)\|\|x_0\| + \int_0^t \|T(t-s)\|\|f(s,x(s))\|ds + \int_0^t \|T(t-s)\|$$
$$\|Q(s,K(s,x(s)))\|ds + \int_0^t [\|T(t-\eta)\|\|AB_2\| + \|T(t-\eta)\|\|B\|$$
$$+\|T(t-\eta)\|\|\sigma B_2\|]\left\|\tilde{W}^{-1}\right\|[\|x_1\| + \|T(b)\|\|x_0\| + \int_0^b \|T(b-s)\|\|f(s,x(s))\|ds$$
$$+\int_0^b \|T(b-s)\|\|Q(s,K(s,x(s)))\|ds]\|\eta\|d\eta$$

$$\|(\Phi x)(t)\| \leq M\|x_0\| + \int_0^t M L_1 ds + \int_0^t M L_2 ds + \int_0^t [M(AB_2) + MC + v(t)]$$
$$K_2[\|x_1\| + M\|x_0\| + \int_0^b M L_1 ds + \int_0^b M L_2 ds]\|\eta\|d\eta.$$ Then

$$\|(\Phi x)(t)\| \leq M\|x_0\| + bM(L_1 + L_2) + [bMAB_2 + bMC + K_1]\,K_2[\|x_1\|$$
$$+M\|x_0\| + bM(L_1 + L_2)]$$

$$\|(\Phi x)(t)\| \leq K_2(bMAB_2 + bMC + K_1)[\|x_1\| + N] + N.$$ Thus $$\|(\Phi x)(t)\| \leq r.$$

Since *f* and Q are continuous and $\|(\Phi x)(t)\| \leq r$, it follows that Φ is also continuous and maps Y_0 into itself.

Now to prove that Φ maps Y_0 into a precompact subset of Y_0, first it shows that for every fixed $t \in J$, the set ; $Y_0(t) = \{(\Phi x)(t) : x \in Y_0\}$ is precompact in *E*. This is clear for $t = 0$, since $Y_0(0) = \{x_0\}$.

Let $t > 0$ is fixed and for $0 < e < t$, define:

$$(\Phi_\epsilon x)(t) = T(t)x_0 + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon) f(s,x(s))\,ds$$
$$+T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)\, Q(s,K(s,x(s)))\,ds$$
$$+T(\epsilon)\int_0^{t-\epsilon} T(t-\eta-\epsilon)\,(AB_2 + B - \sigma B_2)\tilde{W}^{-1}[x_1 - T(b)x_0$$
$$-\int_0^b T(b-s)\; f(s,x(s))\,ds - \int_0^b T(b-s)\; Q\;(s,K(s,x(s)))\,ds\,]\,(\eta)\,d\eta$$

$$(\Phi_\epsilon x)(t) = T(t)x_0 + \int_0^{t-\epsilon} T(t-s) f(s,x(s))\,ds + \int_0^{t-\epsilon} T(t-s)\; Q(s,K(s,x(s)))\,ds$$

$$+\int_0^{t-\epsilon} T(t-\eta)(AB_2 + B - \sigma B_2)\tilde{W}^{-1}[x_1 - T(b)x_0 - \int_0^b T(b-s)\, f(s,x(s))\,ds$$
$$-\int_0^b T(b-s)\; Q(s,K(s,x(s)))\,ds\,](\eta)\,d\eta$$

Since $T(t)$ is compact (hypothesis(2)) for $t > 0$, also the operators $A, B, B_1, \boldsymbol{B}_2, \tilde{W}^{-1}$, f, Q are all bounded ,then

$(\Phi_\epsilon x)(t)$ is totally bounded for every ϵ, $0 < \epsilon < t$, i.e., the set

$$Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Y_0\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$.

Furthermore, for $x \in Y_0$, one has

$$\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \le \int_{t-\epsilon}^{t} \|T(t-s)\, f(s,x(s))\|\,d + \int_{t-\epsilon}^{t} \|T(t-s)\; Q(s,K(s,x(s)))\|\,ds$$
$$+\int_{t-\epsilon}^{t} \Big\| T(t-\eta)(AB_2 + B - \sigma B_2)\,\tilde{W}^{-1}[x_1 - T(b)\,x_0 - \int_0^b T(b-s)\, f(s,x(s))ds$$
$$-\int_0^b T(b-s)\; Q(s,K(s,x(s)))\,ds\,](\eta)\Big\|\,d\eta$$

$$\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \le \int_{t-\epsilon}^{t} \|T(t-s)\|\|f(s,x(s))\|\,ds + \int_{t-\epsilon}^{t} \|T(t-s)\|\|Q(s,K(s,x(s)))\|\,ds$$
$$+\int_{t-\epsilon}^{t} [\|T(t-\eta)\|\|AB_2\| + \|T(t-\eta)\|\|B\| + \|T(t-\eta)\|\|\sigma B_2\|]\,\|\tilde{W}^{-1}\|\,[\|x_1\|$$
$$+\|T(b)\|\|x_0\| + \int_0^b \|T(b-s)\|\|f(s,x(s))\|ds + \int_0^b \|T(b-s)\|$$
$$\|Q(s,K(s,x(s)))\|ds\,]\|\eta\|\,d\eta.$$ And then

$$\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \le \epsilon M(L_1 + L_2) + [\epsilon K_2(MAB_2 + MC) + K_1K_2][\|x_1\| + N]$$

which implies that $Y_0(t)$ is totally bounded, that is precompact in E .[8]

Now, we want to show that

$\Phi(Y_0) = \{\Phi x : x \in Y_0\}$ is an equicontinuous family of functions.

For that, let $t_2>t_1>0$, then one has

$$\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \le \|T(t_1)-T(t_2)\|\|x_0\| + \Big\|\int_0^{t_1}[T(t_1-\eta)-T(t_2-\eta)]\ (AB_2+B-\sigma B_2)$$
$$\tilde{W}^{-1}[x_1-T(b)x_0-\int_0^b T(b-s)\,f(s,x(s))\,ds$$
$$-\int_0^b T(b-s)\,Q(s,K(s,x(s)))\,ds](\eta)d\eta\Big\|$$
$$+\Big\|\int_{t_1}^{t_2} T(t_2-\eta)(AB_2+B-\sigma B_2)\,\tilde{W}^{-1}[x_1-T(b)x_0$$
$$-\int_0^b T(b-s)\,f(s,x(s))\,ds-\int_0^b T(b-s)\,Q(s,K(s,x(s)))\,ds](\eta)d\eta\Big\|$$
$$+\Big\|\int_0^{t_1}[T(t_1-s)-T(t_2-s)][f(s,x(s))+Q(s,K(s,x(s)))]\,ds\Big\|$$
$$+\Big\|\int_{t_1}^{t_2} T(t_2-s)[f(s,x(s))+Q(s,K(s,x(s)))]ds\Big\|$$

$$\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \le \|T(t_1)-T(t_2)\|\|x_0\|+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\|AB_2+B\|\Big\|\tilde{W}^{-1}\Big\|[\|x_1\|+\|T(b)\|\|x_0\|$$
$$+\int_0^b\|T(b-s)\,f(s,x(s))\|\,ds+\int_0^b\|T(b-s)\,Q(s,K(s,x(s)))\|\,ds]\|\eta\|d\eta$$
$$+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\|\sigma B_2\|\Big\|\tilde{W}^{-1}\Big\|[\|x_1\|+\|T(b)\|\|x_0\|$$

$$+\int_0^b\|T(b-s)\,f(s,x(s))\|\,ds+\int_0^b\|T(b-s)\,Q(s,K(s,x(s)))\|\,ds]\|\eta\|d\eta$$
$$+\int_{t_1}^{t_2}\|T(t_2-\eta)\|\|AB_2+B\|\Big\|\tilde{W}^{-1}\Big\|[\|x_1\|+\|T(b)\|\|x_0\|+\int_0^b\|T(b-s)f(s,x(s))\|\,ds$$
$$+\int_0^b\|T(b-s)\,Q(s,K(s,x(s)))\|\,ds]\|\eta\|d\eta+\int_{t_1}^{t_2}\|T(t_2-\eta)\|\|\sigma B_2\|\Big\|\tilde{W}^{-1}\Big\|[\|x_1\|$$

$$+\|T(b)\|\|x_0\|+\int_0^b\|T(b-s)\,f(s,x(s))\|\,ds+\int_0^b\|T(b-s)\,Q(s,K(s,x(s)))\|\,ds]$$

$$\|\eta\|\,d\eta+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\|[f(s,x(s))+Q(s,K(s,x(s)))]\|ds$$

$$+\int_{t_1}^{t_2}\|T(t_2-s)\|\|f(s,x(s))+Q(s,K(s,x(s)))\|\,ds. \text{ Thus}$$

$$\begin{aligned}\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \le{}& \|T(t_1)-T(t_2)\|\|x_0\|+\int_0^{t_1}\|T(t_1-\eta)-T(t_2-\eta)\|(AB_2+C)K_2[\|x_1\| \\ &+M\|x_0\|+\int_0^b M\;L_1\;ds+\int_0^b M\;L_2\;ds]\|\eta\|d\eta+\int_0^{t_1}\|T(t_1-\eta)-T(t_2-\eta)\|\|\sigma B_2\| \\ &K_2[\|x_1\|+M\|x_0\|+bM(L_1+L_2)]\|\eta\|d\eta+\int_{t_1}^{t_2}\|T(t_2-\eta)\|(AB_2+C)K_2[\|x_1\| \\ &+M\|x_0\|+b\;M\;L_1+b\;M\;L_2]\|\eta\|d\eta+\int_{t_1}^{t_2}\|T(t_2-\eta)\|\|\sigma B_2\|K_2[\|x_1\| \\ &+M\|x_0\|+b\;M\;L_1+b\;M\;L_2]\|\eta\|d\eta \\ &+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|(L_1+L_2)\,ds+\int_{t_1}^{t_2}\|T(t_2-s)\|(L_1+L_2)\,ds.\end{aligned}$$

And hence

$$\begin{aligned}\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \le{}& \|T(t_1)-T(t_2)\|\|x_0\|+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|(AB_2+C)K_2 \\ &[\|x_1\|+M\|x_0\|+bM(L_1+L_2)]\,ds+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\| \\ &\|\sigma B_2\|K_2[\|x_1\|+M\|x_0\|+bM(L_1+L_2)]\,ds \\ &+\int_{t_1}^{t_2}\|T(t_2-s)\|(AB_2+C)K_2[\|x_1\|+M\|x_0\|+bM(L_1+L_2)]\,ds \\ &+\int_{t_1}^{t_2}\|T(t_2-s)\|\,\|\sigma B_2\|K_2[\|x_1\|+M\|x_0\|+bM(L_1+L_2)]\,ds \\ &+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\,(L_1+L_2)\,ds+\int_{t_1}^{t_2}\|T(t_2-s)\|(L_1+L_2)\,ds \qquad (10)\end{aligned}$$

Since $T(t)$, $t > 0$ is a compact , then $T(t)$ is continuous in the uniform operator topology for $t > 0$. Thus the right –hand side of (10), which is independent of $x \in Y_0$, tends to zero as $t_2 - t_1 \to 0$, so $\Phi(Y_0)$ is an equicontinuous family of functions.

Also, $\Phi(Y_0)$ is precompact, then by Schauder fixed point theorem,Φ has a fixed point in Y_0 [7] .

Any fixed point of Φ is a mild solution of (1) on J, satisfying

$$(\Phi x)(t) = x(t) \in E.$$

Thus, the system (1) is controllable on *J*.

Remark Assuming that the nonlinear operators *f*, *K*, and *Q* in the problem (1) satisfying Lipschitz condition on the second argument, one can prove that the system (1) controllable on *J* by using Banach fixed point theorem. The results are obtained by showing that the operator Φ in section 3.3, is a contraction mapping, i.e., for

$$x_1(t), x_2(t) \in \Upsilon_0, \quad \|\Phi x_1(t) - \Phi x_2(t)\| \le q\,\|x_1(t) - x_2(t)\|, \qquad 0 \le q < 1.$$

Example: Let ϕ be a bounded and open subset of Υ^n, and let *C* be boundary control integrodifferential system

$$\frac{\partial y(t,x)}{\partial t} - Ny(t,x) = \sigma_1\left(t, y(t,x), \int_0^t \sigma_2\left(t,s,y(s,x)\right)ds\right) \text{ in } Y=(0,b)\times\phi$$
$$y(t,0)= u(t,0)\text{ , on } M=(0,b)\times C\ ,\ t\in[0,b]$$
$$y(0,x)= y_0(x),\ \text{ for } x\in\phi \qquad \ldots(11)$$

where
$u\in L^2(M)$, $y_0\in L^2(\phi)$, $\sigma_1\in L^2(Y)$ and $\sigma_2\in Y$.

The above problem can be formulated as a boundary control problem of the form (1) by suitably taking the spaces, *E*, *X*, *U* and the operators B_1, σ, and *x* as follows:

Let E = $L^2(\emptyset)$, $X = H^{-\frac{1}{2}}(\Gamma)$, U=$L^2(C)$, $B_1 = I$ (the identity operator) and D(σ)={$y \in L^2(\emptyset); Ny \in L^2(\emptyset)$}, σ = N. The operator x is the “trace” operator

such that $xy = y|_C$ is well defined and belongs to $H^{-\frac{1}{2}}(\Gamma)$ for each y∈D(σ) (see [11]) and the operator *A* is given by $A = N, D(A) = H_0^1(\emptyset) \cup H^2(\emptyset)$

(Here $H^k(\emptyset), H^{\propto}(\Gamma)$ and $H_0^1(\emptyset)$ are usual Sobolev Spaces on ∅ , Γ).

Define the linear operator B: $L^2(C) \to L^2(\emptyset)$ by Bu=W_u where W_u is the unique solution to the Dirichlet boundary value problem,

$$DW_u = 0 \text{ in } \emptyset$$
$$W_u = u \text{ in } C$$

In other words (see [12])

$$\int_\phi W_u \Delta W dx = \int_\Gamma u \frac{\partial W}{\partial n} dx\,, \quad \text{for all } W \in H_0^1 \cup H^2(\phi) \qquad (12)$$

where $\frac{\partial W}{\partial n}$ denotes the outward normal derivative of W which is well-defined as an element of $H^{\frac{1}{2}}(\Gamma)$. From (12) it follows that,

$$\|W_u\|_{L^2(\phi)} \le C_1 \|u\|_{H^{-\frac{1}{2}}(\Gamma)}\,, \quad \text{for all } u \in H^{-\frac{1}{2}}(\Gamma)$$

and

$$\| W_u \|_{H^1(\phi)} \leq C_2 \| u \|_{H^{\frac{1}{2}}(\Gamma)}, \quad \text{for all } u \in H^{\frac{1}{2}}(\Gamma),$$

where C_i , $i=1,2$ are positive constants independent of *u.*
From the above estimates it follows by an interpolation argument [14] that

$$\|AT(t)B\|_{L(L^2(\Gamma),L^2(\Gamma))} \leq Ct^{-\frac{3}{4}}, \quad \text{for all } t > 0 \text{ with } v(t) = Ct^{-\frac{3}{4}}$$

Further assume that the bounded invertible operator $\tilde{W}$ exists.
Choose b and other constants, such that satisfying the last condition (9).

Hence, one can see that all the conditions stated in the theorem 3.1, are satisfied and so the system (11) is controllable on [0,b].

Conclusions

1. Generalize nonlinear control problem with boundary condition by taking *f, K, and Q* in system (1) any nonlinear operators, and study the controllability of the problem (1) by using semigroup theory(compact semigroup) and Schauder fixed point theorem .
2. The idea of studying the controllability of problem (1) by using Banach fixed point theorem is introduced.

Future work

The observability and optimality for the problem (1) may be considered.

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