New Class of Meromorphic Univalent Functions

Waggas Galib Atshan and Faiz Jawad Abdulkadhim

Department of Mathematics,

College of Computer Science and Mathematics,

University of Al-Qadisiya ,Diwaniya, Iraq

waggashnd@gmail.com , waggas_hnd@yahoo.com , faiz.jawad@yahoo.com

Abstract: In this paper We define a new class of meromorphic univalent functions in the punctured unit disk. We obtain coefficient inequalities, Closure Theorem, convex companion , distortion bound , Partial sums and neighborhood property, and Hadamard product. Atshan[9], Cho et al.[6], Atshan and Kulkarni[3], Seoudy and Aouf [7] studied the meromorphic univalent functions for another classes.

Keywords: Meromorphic univalent functions , Closure Theorem, convex combination , distortion bound ,Partial sums and neighborhood property

1- **Introduction**: Let $\mathcal{A}_1^*$ be a class of functions f of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Let Σ be a subclass of $\mathcal{A}_1^*$, consisting of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \ , \ \ (a_k \geq 0 \, , k \in \mathbb{N} = \{1,2,\dots\}), \tag{1.2}$$

which are analytic and meromorphic univalent in U^*.

For $f \in \Sigma$ given by (1.2) and $g \in \Sigma$ given by

A function $f \in \Sigma$ is said to be meromorphic starlike function of order ρ ($0 \leq \rho < 1$) if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \rho, \qquad (f(z) \neq 0, z \in U = U^* \cup \{0\}). \tag{1.3}$$

The class of all such functions is defined by $\Sigma^*(\rho)$.

A function $f \in \Sigma$ is said to be meromorphic convex function of order ρ ($0 \leq \rho < 1$) if

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \rho, \qquad (f'(z) \neq 0, z \in U = U^* \cup \{0\}). \tag{1.4}$$

For $f \in \Sigma$ given by (1.2) and $g \in \Sigma$ given by

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \ , \ \ (b_k \geq 0\,, k \in \mathbb{N} = \{1,2,\dots\}), \tag{1.5}$$

the Hadmard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Definition (1): A function $f \in \Sigma$ is said to be in the class $\Sigma\,(\alpha,\gamma)$ if and only if

$$\left|\frac{z(z^2 f'(z))' - z^2 f'(z)}{\gamma - z^2 f'(z)}\right| < \alpha \ \text{, where} \quad (0 < \alpha \leq 1, \gamma > 0). \tag{1.6}$$

Lemma (1)[1]: Let $\alpha \geq 0$. Then, $Re(w) > \alpha$ if and only if $|w-(1+\alpha)| \leq |w+(1-\alpha)|$, where w be any complex number

The first theorem gives a necessary and sufficient condition for a function f to be in the class $\Sigma\,(\alpha,\gamma)$.

2- coefficient bounds:

Theorem (1): Let $f \in \Sigma$. Then $f \in \Sigma\,(\alpha,\gamma)$ if and only if

$$\sum_{n=1}^{\infty} n(n+\alpha)a_n \leq \alpha(1+\gamma), \quad (0<\alpha\leq 1, \gamma>0). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{n(n+\alpha)} z^n, \qquad (n \geq 1)\,. \tag{2.2}$$

Proof: Suppose that the inequality (1.6) holds true and $|z| = 1$. Then, we have

$$\left|z\left(z^2 f'(z)\right)' - z^2 f'(z)\right| - \alpha|\gamma - z^2 f'(z)|$$

$$= \left|\sum_{n=1}^{\infty} n^2 a_n z^{n+1}\right| - \alpha\left|(1+\gamma) - \sum_{n=1}^{\infty} n a_n z^{n+1}\right|$$

$$\leq \sum_{n=1}^{\infty} n^2 a_n |z|^{n+1} - \alpha\,(1+\gamma) + \sum_{n=1}^{\infty} \alpha n a_n |z|^{n+1}$$

$$= \sum_{n=1}^{\infty} n(n+\alpha)a_n - \alpha\,(1+\gamma) \leq 0,$$

by hypothesis. Thus by maximum modulus principle, $f(z) \in \Sigma(\alpha,\gamma)$. To show the converse, suppose that $f(z) \in \Sigma(\alpha,\gamma)$. Then from (1.6), we have

$$\left|\frac{z\left(z^2 f'(z)\right)' - z^2 f'(z)}{\gamma - z^2 f'(z)}\right| = \left|\frac{\sum_{n=1}^{\infty} n^2 a_n z^{n+1}}{(1+\gamma) - \sum_{n=1}^{\infty} n a_n z^{n+1}}\right| < \alpha.$$

Since $Re(z) \leq |z|$ for all z $(z \in U)$, we have

$$Re\left\{\frac{\sum_{n=1}^{\infty} n^2 a_n z^{n+1}}{(1+\gamma) - \sum_{n=1}^{\infty} n a_n z^{n+1}}\right\} \leq \alpha. \tag{2.3}$$

We choose the value of z on the real axis so that $z^2 f'(z)$ is real. Upon clearing the denominator of (2.3) and letting $z \to 1^-$, through real values so we can write (2.3) as

$$\sum_{n=1}^{\infty} n(n+\alpha)a_n \leq \alpha(1+\gamma).$$

Sharpness of the result follows by setting

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{n(n+\alpha)} z^n, \qquad (n \geq 1) .$$

Corollary (3.2.1): Let $f \in \Sigma(\alpha, \gamma)$. Then

$$a_n \leq \frac{\alpha(1+\gamma)}{n(n+\alpha)} \ , \qquad (n \geq 1).$$

3- Convex Linear Combination.

Theorem (2): The class $\Sigma(\alpha, \gamma)$ is closed under convex linear combinations .

Proof: Let f_1 and f_2 be the arbitrary elements of $\Sigma(\alpha, \gamma)$. Then for every t $(0 \leq t \leq 1)$, we show that $(1-t)f_1 + tf_2 \in \Sigma(\alpha, \gamma)$. Thus we have

$$(1-t)f_1 + tf_2 = z^{-1} + \sum_{n=1}^{\infty} [(1-t)a_n + tb_n] z^n \ .$$

Hence,

$$\sum_{n=1}^{\infty} n(n+\alpha)[(1-t)a_n + tb_n]$$

$$= (1-t) \sum_{n=1}^{\infty} n(n+\alpha) + t \sum_{n=1}^{\infty} n(n+\alpha) b_n$$

$$\leq (1-t)\alpha(1+\gamma) + t\alpha(1+\gamma) = \alpha(1+\gamma).$$

This completes the proof.

4. The arithmetic mean

Theorem (3): Let the functions f_k defined by

$$f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,k} \ , \quad (a_{n,k} \geq 0, n \in N \ , k = 1,2,\dots,l)$$

be in the class $\Sigma(\alpha, \gamma)$ for every $k = 1,2,3,\dots,l$, then the function h defined by

$$h(z) = z^{-1} + \sum_{n=1}^{\infty} e_n z^n, \qquad (e_n \geq 0\,, n \in N)$$

also belong to the class $\Sigma(\alpha, \gamma)$, where $e_n = \frac{1}{l}\sum_{n=1}^{\infty} a_{n,k}$.

Proof: Since $f_k \in \Sigma(\alpha, \gamma)$, it follows from Theorem (1) that

$$\sum_{n=1}^{\infty} n(n+\alpha)a_{n,k} \leq \alpha(1+\gamma),$$

for every $k = 1,2,3, \dots, l$. Hence

$$\sum_{n=1}^{\infty} n(n+\alpha)e_n = \sum_{n=1}^{\infty} n(n+\alpha)(\frac{1}{l}\sum_{k=1}^{l} a_{n,k})$$

$$= \frac{1}{l}\sum_{k=1}^{l}\left(\sum_{n=1}^{\infty} n(n+\alpha)a_{n,k}\right) \leq \frac{1}{l}\sum_{k=1}^{l} \alpha(1+\gamma) = \alpha(1+\gamma).$$

Then $h \in \Sigma(\alpha, \gamma)$.

5- The growth and distortion bounds for the functions in the class $\Sigma(\alpha, \gamma, \lambda)$

Theorem (4): If $f \in \Sigma(\alpha, \gamma)$, then

$$\frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\alpha)}r \leq |f(z)| \leq \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\alpha)}r\,, (|z| = r < 1). \qquad (5.1)$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{(1+\alpha)}z. \qquad (5.2)$$

Proof: Let $f \in \Sigma(\alpha, \gamma)$. Then by Theorem (1), we get

$$(1+\alpha)\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} n(n+\alpha)a_n \leq \alpha(1+\gamma)$$

or

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha(1+\gamma)}{(1+\alpha)}. \tag{5.3}$$

Hence,

$$|f(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n$$

$$= \frac{1}{r} + r \sum_{n=1}^{\infty} a_n$$

$$\leq \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\alpha)} r\,. \tag{5.4}$$

Similarly,

$$|f(z)| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n$$

$$= \frac{1}{r} - r \sum_{n=1}^{\infty} a_n$$

$$\geq \frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\alpha)} r. \tag{5.5}$$

From (5.4) and (5.5), we get (5.1) and the proof is complete.

Theorem (5): If $f \in \Sigma(\alpha, \gamma)$, then

$$\frac{1}{r^2} - \frac{\alpha(1+\gamma)}{(1+\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1+\gamma)}{(1+\alpha)} \quad (|z| = r < 1). \tag{5.6}$$

The result is sharp for the function f is given by (2.2).

Proof: The proof is similar to that of Theorem (4).

6. The Partial sums and neighborhood property

Theorem (6): Let $f \in \Sigma$ be given by (1.2) and define the partial sums $S_1(z)$ and $S_k(z)$ as follows $S_1(z) = z^{-1}$ and

$$S_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n, \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{6.1}$$

Also suppose that

$$\sum_{n=1}^{\infty} c_n a_n \le 1, \quad \left(c_n = \frac{n(n+\alpha)}{\alpha(1+\gamma)}\right). \tag{6.2}$$

Then, we have

$$Re\left\{\frac{f(z)}{S_k(z)}\right\} > 1 - \frac{1}{c_k}, \qquad (z \in U, k \in \mathbb{N}) \tag{6.3}$$

and

$$Re\left\{\frac{S_k(z)}{f(z)}\right\} > \frac{c_k}{1+c_k}, \qquad (z \in U, k \in \mathbb{N}). \tag{6.4}$$

Each of the bounds in (6.3) and (6.4) is the best possible for $n \in \mathbb{N}$.

Proof: We can see from (6.2) that $c_{n+1} > c_n > 1 \ , n = 1,2,3 \ldots$. Therefore, we have

$$\sum_{n=1}^{k-1} a_n + c_k \sum_{n=k}^{\infty} a_n \le \sum_{n=1}^{\infty} c_n a_n \le 1. \tag{6.5}$$

By setting

$$g_1(z) = c_k \left[\frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{c_k}\right)\right] = 1 + \frac{c_k \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}}, \tag{6.6}$$

and applying (6.5), we find that

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \le \frac{c_k \sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n - c_k \sum_{k=n}^{\infty} a_n}, \tag{6.7}$$

which readily yields the assertion (6.3). If we take

$$f(z) = z^{-1} - \frac{z^k}{c_k}, \qquad (6.7)$$

then

$$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{c_k} \longrightarrow 1 - \frac{1}{c_k} \;\; (\mathrm{z} \longrightarrow 1^-),$$

which shows that the bound in (6.3) is the best possible for $k \in \mathbb{N}$.

Similarly, if we put

$$g_2(z) = (1 + c_k)\left[\frac{S_k(z)}{f(z)} - \frac{c_k}{1 + c_k}\right] = 1 - \frac{(1 + c_k)\sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{k+1}}, \qquad (6.8)$$

and make use of (6.8), we have

$$\left|\frac{g_2(z) - 1}{g_2(z) + 1}\right| \leq \frac{(1 + c_k)\sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n + (1 - c_k)\sum_{n=k}^{\infty} a_n}, \qquad (6.9)$$

which leads us to the assertion (6.4). The bound in (6.5) is sharp for each $k \in \mathbb{N}$ with the function given by (6.7). The proof of the theorem is complete.

Now, following the earlier works on neighborhoods of analytic functions by Goodman [4] and Ruscheweyh [6], we begin by introducing here the δ −neighborhood of a function $f \in \Sigma$ of the form (1.2) by means of the definition below:

$$N_\delta(f) = \left\{g \in \Sigma : g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \;\; and \;\; \sum_{k=1}^{\infty} k|a_k - b_k| \leq \delta\,, 0 \leq \delta < 1\right\}. \qquad (6.10)$$

Particularly for the identity function $e(z) = z^{-1}$, we have

$$N_\delta(e) = \left\{g \in \Sigma : g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \;\; and \;\; \sum_{k=1}^{\infty} k|b_k| \leq \delta, 0 \leq \delta < 1\right\}. \qquad (6.11)$$

Definition (2): A function $f \in \mathcal{M}$ is said to be in the class $\Sigma_y(\alpha, \gamma)$ if there exists a function $g \in \Sigma(\alpha, \gamma)$, such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - y, \;\; (z \in U, 0 \leq y < 1).$$

Theorem (7): If $g \in \Sigma(\alpha,\gamma)$ and

$$y = 1 - \frac{\delta(1+\alpha)}{1-\alpha\gamma}, \qquad (6.12)$$

then $N_\delta(g) \subset \Sigma_{\boldsymbol{y}}(\alpha,\gamma)$.

Proof: Let $f \in N_\delta(g)$. Then we find from (6.10)that

$$\sum_{k=1}^{\infty} k|a_k - b_k| \le \delta,$$

which implies the coefficient inequality

$$\sum_{k=1}^{\infty} |a_k - b_k| \le \delta, \qquad (k \in \mathbb{N}).$$

Since $g \in \Sigma(\alpha,\gamma)$, then by using Theorem (1), we get

$$\sum_{n=1}^{\infty} b_n \le \frac{\alpha(1+\gamma)}{(1+\alpha)},$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{n=1}^{\infty}|a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \le \frac{\delta(1+\alpha)}{1-\alpha\gamma} = 1 - y.$$

Hence, by Definition (2), $f \in \Sigma_y(\alpha,\gamma)$ for y given by (6.12).

This completes the proof.

7. the radii of starlikeness and convexity .

Theorem (8): If $f \in \Sigma(\alpha,\gamma)$, then f is univalent meromorphic starlike of order φ $(0 \le \varphi < 1)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_k \left\{\frac{n(1-\varphi)(n+\alpha)}{(n-\varphi+2)\alpha(1+\gamma)}\right\}^{\frac{1}{n+1}}.$$

The result is sharp for the function f given by (2.2).

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\varphi \quad \text{for } |z|<r_1. \tag{7.1}$$

But

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{zf'(z)+f(z)}{f(z)}\right| \leq \frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}}.$$

Thus, (7.1) will be satisfied if

$$\frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}} \leq 1-\varphi,$$

or if

$$\sum_{n=1}^{\infty}\frac{(n-\varphi+2)}{1-\varphi}a_n|z|^{n+1} \leq 1. \tag{7.2}$$

Since $f \in \Sigma(\alpha,\gamma)$, we have

$$\sum_{n=1}^{\infty}\frac{n(n+\alpha)}{\alpha(1+\gamma)}a_n \leq 1.$$

Hence, (7.2) will be true if

$$\frac{(n-\varphi+2)}{1-\varphi}|z|^{k+1} \leq \frac{n(n+\alpha)}{\alpha(1+\gamma)},$$

or equivalently

$$|z| \leq \left\{\frac{n(1-\varphi)(n+\alpha)}{(n-\varphi+2)\alpha(1+\gamma)}\right\}^{\frac{1}{n+1}}, \qquad (n \geq 1),$$

which follows the result.

Theorem (9): If $f \in \Sigma(\alpha,\gamma)$, then f is univalent meromorphic convex of order φ $(0 \leq \varphi < 1)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_n\left\{\frac{(1-\varphi)(n+\alpha)}{\alpha(n-\varphi+2)(1+\gamma)}\right\}^{\frac{1}{n+1}}.$$

The result is sharp for the function f given by (2.2).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\varphi \quad \text{for } |z|<r_2. \tag{7.3}$$

But

$$\left|\frac{zf''(z)}{f'(z)}+2\right| = \left|\frac{zf''(z)+2f'(z)}{f'(z)}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_k|z|^{n+1}}{1-\sum_{n=1}^{\infty} na_n|z|^{n+1}}.$$

Thus, (7.3) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty} na_n|z|^{n+1}} \leq 1-\varphi,$$

or if

$$\sum_{n=1}^{\infty} \frac{n(n-\varphi+2)}{1-\varphi} a_n|z|^{n+1} \leq 1. \tag{7.4}$$

Since $f \in \Sigma(\alpha,\gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{n(n+\alpha)}{\alpha(1+\gamma)} a_n \leq 1.$$

Hence, (7.4) will be true if

$$\frac{n(n-\varphi+2)}{1-\varphi}|z|^{n+1} \leq \frac{n(n+\alpha)}{\alpha(1+\gamma)},$$

or equivalently

$$|z| \leq \left\{\frac{(1-\varphi)(n+\alpha)}{\alpha(n-\varphi+2)(1+\gamma)}\right\}^{\frac{1}{n+1}}, \qquad (n \geq 1),$$

which follows the result.

8. the convolution properties .

Theorem (10) : Let the functions f and g of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \ , \ g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$$

belong to the class $\Sigma(\alpha,\gamma)$. Then the Hadamard product (or convolution) of two functions f and g belong to the class $\Sigma(\alpha,\gamma_1)$, where

$$\gamma_1 = \frac{\alpha}{n(n+\alpha)} - 1.$$

Proof: Since f and $g \in \Sigma(\alpha,\gamma)$. Then from Theorem (2.1), we have

$$\sum_{n=1}^{\infty} \frac{n(n+\alpha)}{\alpha(1+\gamma)} a_n \leq 1 \quad , \qquad \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{\alpha(1+\gamma)} b_n \leq 1.$$

We need to find largest number γ_1 such that

$$\sum_{n=1}^{\infty} \frac{n(n+\alpha)}{\alpha(1+\gamma_1)} a_n . b_n \leq 1.$$

By using Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{n(n+\alpha)}{\alpha(1+\gamma)} \sqrt{a_n . b_n} \leq 1. \tag{8.1}$$

Thus, it is sufficient to show that

$$\frac{n(n+\alpha)}{\alpha(1+\gamma_1)} a_n . b_n \leq \frac{n(n+\alpha)}{\alpha(1+\gamma)} \sqrt{a_n . b_n} \, .$$

This is equivalent to

$$\sqrt{a_n . b_n} \leq \frac{1+\gamma_1}{1+\gamma}. \tag{8.2}$$

From (8.1), we get

$$\sqrt{a_n . b_n} \leq \frac{\alpha(1+\gamma)}{n(n+\alpha)}.$$

Thus, it is sufficient to show that

$$\frac{\alpha(1+\gamma)}{n(n+\alpha)} \leq \frac{1+\gamma_1}{1+\gamma},$$

which implies

$$\gamma_1 \geq \frac{\alpha}{n(n+\alpha)} - 1.$$

References:

[1] E. S. Aqlan, Some Problems Connected with Geometric Function Theory, Ph. D. Thesis (2004), Pune University, Pune.

[2] W. G. Atshan, Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II, J. Surveys in Mathematics and its Applications, 3 (2008), 67-77.

[3] W. G. Atshan and S. R. Kulkarni, Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative I, J. Rajasthan Acad. Phys. Sci. 6(2)(2007), 129-140

[4] A. W. Goodman, Univalent functions and non-analytic curves, Proc. Amer. Math. Soc. , 8(1975), 598-601.

[5] N. E. Cho, H. Lee and S. Owa, A class of meromorpgic univalent functions with positive coefficients, Koebe Math. , 4(1987), 43-50.

[6] S. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc. , 81(1981), 521-527.

[7] T. M. Seoudy and M. K. Aouf, Some inequalities of meromorphic p-valent functions associated wuth the Liu-Srivastava operator, Proceedings of IAM, 2(1) (2013), 1-9.