New Subclass of P-valent Functions Defined by Convolution with Negative Coefficients

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Abstract

In this paper, we introduce and study new subclass $f(z) \in \Im S_g(p, v, \beta, \lambda)$ defined by convolution. we obtain necessary and sufficient condition of these class and some properties (the radii of starlikeness, convexity and close- to- convexity; weighted mean and arithmetic mean; We also obtain convolution properties and neighborhood property of the functions $f(z)$ in this class).

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1. Introduction

Let $\mathcal{A}_p(n)$ be the class of normalized functions f of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k , \qquad (p, n \in \mathrm{N} = \{1,2, \dots\}, z \in U), \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.
Let $\Im_p(n)$ be the subclass of $\mathcal{A}_p(n)$ consisting functions f of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k , \quad (a_k \geq 0, p, n \in \mathrm{N} = \{1,2, \dots\}, z \in U), \tag{2}$$

which are analytic and p-valent in U.

Let (f*g)(z) denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1) and $g(z)$ is given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k, \qquad (z \in U) \tag{3}$$

then

$$(f * g) = f(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k. \ (z \in U) \tag{4}$$

In this paper, we will use (1) to define a new subclass of $\Im_p(n)$ as follows:

$\Im S_g(p, v, \beta, \lambda) = \{ f(z) \in \Im_p(n)$:

$$\left| \frac{\frac{z(f*g)'(z)}{p(f*g)(z)} - \alpha \left| \frac{z(f*g)'(z)}{p(f*g)(z)} - 1 \right| - p}{\beta(p-\lambda) - \frac{z(f*g)'(z)}{p(f*g)(z)} - \alpha \left| \frac{z(f*g)'(z)}{p(f*g)(z)} - 1 \right|} \right| < v,$$

$$0 < \beta \le 1, 0 \le \lambda < p, \alpha \ge 0, 0 < v \le 1, p, n \in \mathrm{N}, z \in U\}. \tag{5}$$

Some authors studied for another classes, like, Atshan, Mustafa and Mouajeeb [1], Aouf and Mostafa[2], Mahzoon[9]and Yang and Li [14] consisting of multivalent functions.

2. Necessary and Sufficient Condition for $f(z) \in IS_g(p, v, \beta, \lambda)$

Theorem 1. Let the function $f(z) \in \Im_p(n)$ be given by (1), then $f(z) \in IS_g(p, v, \beta, \lambda)$ if and only if

$$\sum_{k=n+p}^{\infty} [(k-p)(\alpha(1+v)+1) - v(k - \beta p(p-\lambda))] a_k b_k \le vp(\beta p - \beta\lambda - 1), \tag{6}$$

where $0 < \beta \le 1, 0 \le \lambda < p, \alpha \ge 0, 0 \le \lambda < p, p, n \in N, z \in U$.

Proof. Suppose that the equality(6)holds true and let $|z| = 1$

$$\left| \frac{\frac{z(f*g)'(z)}{p(f*g)(z)} - \alpha \left| \frac{z(f*g)'(z)}{p(f*g)(z)} - 1 \right| - p}{\beta(p-\lambda) - \frac{z(f*g)'(z)}{p(f*g)(z)} - \alpha \left| \frac{z(f*g)'(z)}{p(f*g)(z)} - 1 \right|} \right| < v,$$

$$= \left| \frac{z(f*g)'(z) - \alpha|z(f*g)'(z) - p(f*g)(z)| - p(f*g)(z)}{\beta p(p-\lambda)(f*g)(z) - z(f*g)'(z) - \alpha|z(f*g)'(z) - p(f*g)(z)|} \right| < v,$$

then

$$|z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|-p(f*g)(z)|$$

$$-v|\beta p(p-\lambda)(f*g)(z)-z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|$$

$$=\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k-\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|\right|$$

$$-v\left|(\beta p^2-\beta p\lambda-p)z^p-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k]a_kb_kz^k-\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|\right|$$

$$\leq\sum_{k=n+p}^{\infty}(k-p)a_kb_k|z|^k+\alpha\sum_{k=n+p}^{\infty}(k-p)a_kb_k|z|^k$$

$$+\sum_{k=n+p}^{\infty}v[\beta p(p-\lambda)-k]a_kb_k|z|^k+\alpha v\sum_{k=n+p}^{\infty}(k-p)a_kb_k|z|^k-v(\beta p^2-\beta p\lambda-p)|z|^p$$

$$=\sum_{k=n+p}^{\infty}[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]a_kb_k-vp(\beta p-\beta\lambda-1)\leq 0.$$

Hence, by the maximum modulus Theorem, for any $z\in U$, we have

$$|z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|-p(f*g)(z)|$$

$$-v|\beta p(p-\lambda)(f*g)(z)-z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|$$

$$\leq\sum_{k=n+p}^{\infty}[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]a_kb_k-vp(\beta p-\beta\lambda-1)\leq 0,$$

which is equivalent to

$$\sum_{k=n+p}^{\infty}[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]a_kb_k\leq vp(\beta p-\beta\lambda-1).$$

Then $f(z)\in\Im S_g(p,v,\beta,\lambda)$.

Conversely, assume that $f(z)\in\Im S_g(p,v,\beta,\lambda)$. Then from (5), we have

$$\left|\frac{\frac{z(f*g)'(z)}{p(f*g)(z)}-\alpha\left|\frac{z(f*g)'(z)}{p(f*g)(z)}-1\right|-p}{\beta(p-\lambda)-\frac{z(f*g)'(z)}{p(f*g)(z)}-\alpha\left|\frac{z(f*g)'(z)}{p(f*g)(z)}-1\right|}\right|<v,$$

$$=\left|\frac{z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|-p(f*g)(z)}{\beta p(p-\lambda)(f*g)(z)-z(f*g)'(z)-\alpha|z(f*g)'(z)-p(f*g)(z)|}\right|$$

$$\left|\frac{-\left(\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k+\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|\right)}{(\beta p^2-\beta p\lambda-p)z^p-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k](k-p)a_kb_kz^k-\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|}\right|$$

$< v,$

since $Re(z) < |z|$ for all z, we have

$$Re\left\{\frac{\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k+\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|}{(\beta p^2-\beta p\lambda-p)z^p-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k](k-p)a_kb_kz^k-\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|}\right\}$$

$< v.$

We choose the values of z on the real axis, so that $\frac{z(f*g)'(z)}{(f*g)(z)}$ is real. Then , we have

$$\left|\frac{\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k+\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|}{(\beta p^2-\beta p\lambda-p)z^p-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k](k-p)a_kb_kz^k-\alpha\left|-\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k\right|}\right|$$

$$=\left|\frac{\sum_{k=n+p}^{\infty}(k-p)a_kb_kz^k+\alpha\sum_{k=n+p}^{\infty}(k-p)a_kb_k|z|^k}{(\beta p^2-\beta p\lambda-p)z^p-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k](k-p)a_kb_kz^k-\alpha\sum_{k=n+p}^{\infty}(k-p)a_kb_k|z|^k}\right|< v.$$

(7)

Letting $z \to 1^-$throughout real values in (7), we obtain

$$\frac{\sum_{k=n+p}^{\infty}(k-p)a_kb_k+\alpha\sum_{k=n+p}^{\infty}(k-p)a_kb_k}{(\beta p^2-\beta p\lambda-p)-\sum_{k=n+p}^{\infty}[\beta p(p-\lambda)-k](k-p)a_kb_k-\alpha\sum_{k=n+p}^{\infty}(k-p)a_kb_k}< v,$$

it is

$$\sum_{k=n+p}^{\infty}[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]a_kb_k \le vp(\beta p-\beta\lambda-1).$$

This completes the proof of the theorem.

Corollary 1. Let the function $f(z) \in \Im_p(n)$be given by (1). If $f(z) \in \Im S_g(p,v,\beta,\lambda)$, then

$$a_k \leq \frac{vp(\beta p - \beta\lambda - 1)}{\left[(k-p)(\alpha(1+v)+1) - v\left(k - \beta p(p-\lambda)\right)\right]b_k}.$$

The result is sharp for the function given by

$$f(z) = z^p - \frac{vp(\beta p - \beta\lambda - 1)}{[(k-p)(\alpha(1+v)+1) - v(k - \beta p(p-\lambda))]b_k} z^k.$$

3. Radii of Starlikeness, Convexity and Close-to-Convexity

In the section, we obtain the radii of starlikeness, convexity and close − to − convexity for functions in the class $\Im S_g(p, v, \beta, \lambda)$.

Theorem 2. Let the function $f(z)$defined by (1) be in the class $\Im S_g(p, v, \beta, \lambda)$. Then $f(z)$ is starlike of order $\delta(0 \leq \delta < p)$ in $|z| < R_1(k, p, v, \lambda, \beta, \delta)$, where

$$R_1(k, p, v, \lambda, \beta, \delta) = inf\left\{\frac{(p-\delta)\left[(k-p)(\alpha(1+v)+1) - v\left(k - \beta p(p-\lambda)\right)\right]b_k}{vp(k-\delta)(\beta p - \beta\lambda - 1)}\right\}^{1/k-p} \tag{8}$$

Proof. We need to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta \qquad \text{for } |z| < R_1(k, p, v, \lambda, \beta, \delta).$$

Since

$$\left|\frac{zf'(z) - pf(z)}{f(z)}\right| = \left|\frac{-\sum_{k=n+p}^{\infty}(k-p)a_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k}\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} a_k|z|^{k-p}}.$$

To prove (8), it is sufficient to prove

$$\frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} a_k|z|^{k-p}} \leq p - \delta.$$

It is equivalent to

$$\sum_{k=n+p}^{\infty} \frac{(k-p)}{(p-\delta)} a_k|z|^{k-p} \leq 1. \tag{9}$$

By Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1) - v(k - \beta p(p-\lambda))]}{vp(\beta p - \beta\lambda - 1)} a_k b_k \leq 1,$$

hence (9) will be true if

$$\frac{(k-\delta)}{(p-\delta)}|z|^{k-p} \le \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(\beta p-\beta\lambda-1)}$$

It is equivalent to

$$|z|^{k-p} \le \frac{(p-\delta)[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(k-\delta)(\beta p-\beta\lambda-1)},$$

therefore

$$|z| \le \left\{\frac{(p-\delta)[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(k-\delta)(\beta p-\beta\lambda-1)}\right\}^{1/k-p}.$$

This completes the proof .

Theorem 3. Let the function $f(z)$defined by (1) be in the class $\Im S_g(p,v,\beta,\lambda)$. Then $f(z)$ is convex of order $\eta(0 \le \delta < p)$ in $|z| < R_2(k,p,v,\lambda,\beta,\delta)$, where

$$R_2(k,p,v,\lambda,\beta,\delta) = inf\left\{\frac{(p-\delta)[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{v(k^2-k\delta)(\beta p-\beta\lambda-1)}\right\}^{1/k-p} \tag{10}$$

Proof. We need to show that

$$\left|\frac{zf''(z)}{f'(z)}-(p-1)\right| \le p-\delta \qquad \text{for } |z| < R_2(k,p,v,\lambda,\beta,\delta).$$

Since

$$\left|\frac{zf''(z)-(p-1)f'(z)}{f'(z)}\right| = \left|\frac{-\sum_{k=n+p}^{\infty} k(k-p)a_k z^{k-1}}{pz^{p-1}-\sum_{k=n+p}^{\infty} ka_k z^{k-1}}\right| \le \frac{\sum_{k=n+p}^{\infty} k(k-p)a_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty} ka_k|z|^{k-p}},$$

to prove (10), it is sufficient to prove

$$\frac{\sum_{k=n+p}^{\infty} k(k-p)a_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty} ka_k|z|^{k-p}} \le p-\delta.$$

It is equivalent to

$$\sum_{k=n+p}^{\infty}\frac{k(k-\delta)}{p(p-\delta)}a_k|z|^{k-p} \le 1. \tag{11}$$

By Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]}{vp(\beta p-\beta\lambda-1)} a_k b_k \leq 1,$$

hence (11) will be true if

$$\frac{k(k-\delta)}{p(p-\delta)}|z|^{k-p} \leq \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(\beta p-\beta\lambda-1)}$$

It is equivalent to

$$|z|^{k-p} \leq \frac{(p-\delta)[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{v(k^2-k\delta)(\beta p-\beta\lambda-1)},$$

therefore

$$|z| \leq \left\{\frac{(p-\delta)[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{v(k^2-k\delta)(\beta p-\beta\lambda-1)}\right\}^{1/k-p}.$$

This completes the proof .

Theorem 4. Let the function $f(z)$defined by (1) be in the class $\Im S_g(p,v,\beta,\lambda)$. Then $f(z)$ is close– to– convex of order $\mu(0 \leq \delta < p)$ in $|z| < R_3(k,p,v,\lambda,\beta,\delta)$, where

$$R_3(k,p,v,\lambda,\beta,\delta) = inf\left\{\frac{(p-\delta)\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]b_k}{vpk(\beta p-\beta\lambda-1)}\right\}^{1/k-p} \tag{12}$$

Proof. We need to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\delta \qquad \text{for } |z| < R_3(k,p,v,\lambda,\beta,\delta).$$

Since

$$\left|\frac{f'(z)-pz^{p-1}}{z^{p-1}}\right| = \left|\frac{-\sum_{k=n+p}^{\infty} ka_k z^{k-1}}{z^{p-1}}\right| \leq \sum_{k=n+p}^{\infty} ka_k|z|^{k-p},$$

to prove (12), it is sufficient to prove

$$\sum_{k=n+p}^{\infty} ka_k|z|^{k-p} \leq p-\delta.$$

It is equivalent to

$$\sum_{k=n+p}^{\infty} \frac{k}{(p-\delta)} a_k|z|^{k-p} \leq 1. \tag{13}$$

By Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]}{vp(\beta p-\beta\lambda-1)} a_k b_k \leq 1,$$

hence (13) will be true if

$$\frac{k}{(p-\delta)}|z|^{k-p} \leq \frac{\left[(k-p)(\alpha(1+v)+1)-v\left(k-\beta p(p-\lambda)\right)\right]b_k}{vp(\beta p-\beta\lambda-1)}.$$

It is equivalent to

$$|z|^{k-p} \leq \frac{(p-\delta)\left[(k-p)(\alpha(1+v)+1)-v\left(k-\beta p(p-\lambda)\right)\right]b_k}{vpk(\beta p-\beta\lambda-1)},$$

therefore

$$|z|^{k-p} \leq \left\{\frac{(p-\delta)\left[(k-p)(\alpha(1+v)+1)-v\left(k-\beta p(p-\lambda)\right)\right]b_k}{vpk(\beta p-\beta\lambda-1)}\right\}^{1/k-p}.$$

This completes the proof .

4. Weighted Mean and Arithmetic Mean

Definition 1. Let the function $f_t(z)(t=1,2)$ defined by

$$f_t(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,t} z^k, (t=1,2) \tag{12}$$

belong to $\Im S_g(p,v,\beta,\lambda)$, then the weighted mean $h_j(z)$ of $f_t(z)(t=1,2)$ is given by

$$h_j(z) = \frac{1}{2}[(1-j)f_1(z)+(1+j)f_2(z)].$$

In the theorem below we will show the weighted mean for this class.

Theorem 5. If $f_t(z)(t=1,2)$ are in the class $\Im S_g(p,v,\beta,\lambda)$, then the weighted mean of $f_t(z)(t=1,2)$ is also in $\Im S_g(p,v,\beta,\lambda)$.

Proof. We have $h_j(z)$ by Definition 1,

$$h_j(z) = \frac{1}{2}\left[(1-j)\left(z^p - \sum_{k=n+p}^{\infty} a_{k,1} z^k\right) + (1+j)\left(z^p - \sum_{k=n+p}^{\infty} a_{k,2} z^k\right)\right]$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{1}{2}\left[(1-j)a_{k,1} + (1+j)a_{k,2}\right] z^k.$$

Since $f_t(z)(t=1,2)$ are in the class $\Im S_g(p,v,\beta,\lambda)$ so by Theorem 1, we must prove that

$$\sum_{k=n+p}^{\infty}\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]\frac{1}{2}\left[(1-j)a_{k,1}+(1+j)a_{k,2}\right]b_k$$

$$=\frac{1}{2}(1-j)\sum_{k=n+p}^{\infty}\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]a_{k,1}\,b_k$$

$$+\frac{1}{2}(1+j)\sum_{k=n+p}^{\infty}\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]a_{k,2}\,b_k$$

$$\leq \tfrac{1}{2}(1-j)vp(\beta p-\beta\lambda-1)+\tfrac{1}{2}(1+j)vp(\beta p-\beta\lambda-1)$$

$$=vp(\beta p-\beta\lambda-1).$$

which shows that $h_j(z)\in\Im S_g(p,v,\beta,\lambda)$.
The proof is complete.

Theorem 6. Let the functions $f_i(z)$ defined by

$$f_i(z)=z^p-\sum_{k=n+p}^{\infty}a_{k,i}\,z^k\quad,\big(a_{k,i}\geq 0,\;\; i=1,2,\ldots\ell\big) \tag{13}$$

be in the class $\Im S_g(p,v,\beta,\lambda)$, then arithmetic mean of $f_i(z)(i=1,2,\ldots\ell)$ defined by

$$\mathrm{H}(z)=\frac{1}{\ell}\sum_{i=1}^{\ell}f_i(z), \tag{14}$$

Is also in the class $\Im S_g(p,v,\beta,\lambda)$.

proof . By (13),(14) we can write

$$\mathrm{H}(z)=\frac{1}{\ell}\sum_{i=1}^{\ell}\left(z^p-\sum_{k=n+p}^{\infty}a_{k,i}\,z^k\right)=z^p-\sum_{k=n+p}^{\infty}\left(\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}\right)z^k$$

Since $f_i(z)\in\Im S_g(p,v,\beta,\lambda)$ for every $i=1,2,\ldots\ell$, so by using Theorem 1, we prove that

$$\sum_{k=n+p}^{\infty}[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]\left(\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}\right)b_k$$

$$=\frac{1}{\ell}\sum_{i=1}^{\ell}\left(\sum_{k=n+p}^{\infty}\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]a_{k,i}\,b_k\right)$$

$$\leq vp(\beta p-\beta\lambda-1).$$

which shows that H(z) $\in\Im S_g(p,v,\beta,\lambda)$.
The proof is complete.

5. Convolution Properties.

Theorem 7. If $f_t(z)(t=1,2)$ defined by (12) be in the class $\Im S_g(p,v,\beta,\lambda)$. ThenThe Hadamard product of the functions $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1}\, a_{k,2}\, z^k, \qquad (15)$$

is in the class $\Im S_g(p,v,\beta_1,\lambda)$,where
$\beta_1 \leq$

$$\frac{vp(\beta p-\beta\lambda-1)^2[vk-(k-p)(\alpha(1+v)+1)]-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k}{(p-\lambda)\left[v^2p^2(\beta p-\beta\lambda-1)-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k\right]}.$$

Proof. We need to find the largest β_1 such that

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta_1 p(p-\lambda))]b_k}{vp(\beta_1 p-\beta_1\lambda-1)} a_{k,1}\, a_{k,2} \leq 1.$$

Since the functions $f_t(z)(t=1,2)$ belong to class $G_w^+(\beta,\lambda,\alpha)$, then from Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(\beta p-\beta\lambda-1)} a_{k,t} \leq 1, (t=1,2)$$

by the Cauchy – Schwarzinequality, we have

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{vp(\beta p-\beta\lambda-1)} \sqrt{a_{k,1}\, a_{k,2}} \leq 1. \qquad (16)$$

Thus, we want only to show that

$$\frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta_1 p(p-\lambda))]}{(\beta_1 p-\beta_1\lambda-1)} a_{k,1}\, a_{k,2}$$
$$\leq \frac{\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]}{(\beta p-\beta\lambda-1)} \sqrt{a_{k,1}\, a_{k,2}}.$$

That is, if

$$\sqrt{a_{k,1}\, a_{k,2}} \leq \frac{(\beta_1 p-\beta_1\lambda-1)\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]}{(\beta p-\beta\lambda-1)[(k-p)(\alpha(1+v)+1)-v(k-\beta_1 p(p-\lambda))]},$$

from (19), we have

$$\sqrt{a_{k,1}\, a_{k,2}} \leq \frac{vp(\beta p-\beta\lambda-1)}{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}.$$

Consequently, if

$$\frac{vp(\beta p-\beta\lambda-1)}{\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]b_k}$$

$$\leq \frac{(\beta_1 p-\beta_1\lambda-1)\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]}{(\beta p-\beta\lambda-1)[(k-p)(\alpha(1+v)+1)-v(k-\beta_1 p(p-\lambda))]}. \tag{17}$$

From (17), we have

$$\beta_1 \leq$$

$$\frac{vp(\beta p-\beta\lambda-1)^2[vk-(k-p)(\alpha(1+v)+1)]-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k}{(p-\lambda)\left[v^2p^2(\beta p-\beta\lambda-1)-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k\right]}.$$

This completes the proof of Theorem 7.

Theorem 8. Let the functions $f_t(z)\ (t=1,2)$ defined by (12) be in the class $\Im S_g(p,v,\beta,\lambda)$. Then the function

$$F(z)=z^p-\sum_{k=n+p}^{\infty}\left(a_{k,1}{}^2+a_{k,2}{}^2\right)z^k, \tag{18}$$

also belong to the class $\Im S_g(p,v,\gamma,\lambda)$, where

$$\gamma \leq$$

$$\frac{2vp(\beta p-\beta\lambda-1)^2[vk-(k-p)(\alpha(1+v)+1)]-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k}{(p-\lambda)\left[2v^2p^2(\beta p-\beta\lambda-1)^2-\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k\right]}.$$

proof. By Theorem 1, we want to find the largest γ such that

$$\sum_{k=n+p}^{\infty}\frac{[(k-p)(\alpha(1+v)+1)-v(k-\gamma p(p-\lambda))]b_k}{(\gamma p-\gamma\lambda-1)}\left(a_{k,1}{}^2+a_{k,2}{}^2\right)\leq 1.$$

Since $f_t(z)(t=1,2)$belong to the class $\Im S_g(p,v,\beta,\lambda)$, we have

$$\sum_{k=n+p}^{\infty}\frac{\left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k{}^2}{(\beta p-\beta\lambda-1)^2}a_{k,1}{}^2$$

$$\leq\left(\sum_{k=n+p}^{\infty}\frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]b_k}{(\beta p-\beta\lambda-1)}a_{k,1}{}^2\right)^2\leq 1,$$

and

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]^2 {b_k}^2}{(\beta p-\beta\lambda-1)^2} {a_{k,2}}^2$$

$$\leq \left(\sum_{k=n+p}^{\infty} \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))] b_k}{(\beta p-\beta\lambda-1)} {a_{k,2}}^2 \right)^2 \leq 1.$$

Hence, we have

$$\sum_{k=n+p}^{\infty} \frac{1}{2} \left(\frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]^2 {b_k}^2}{(\beta p-\beta\lambda-1)^2} \right) \left({a_{k,1}}^2 + {a_{k,2}}^2 \right) \leq 1,$$

$F(z) \in \Im S_g(p, v, \gamma, \lambda)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{\left[(k-p)(\alpha(1+v)+1)-v\big(k-\gamma p(p-\lambda)\big)\right] b_k}{(\gamma p-\gamma\lambda-1)} \left({a_{k,1}}^2 + {a_{k,2}}^2 \right) \leq 1.$$

Therefore, we need to find the largest γ such that

$$\frac{\left[(k-p)(\alpha(1+v)+1)-v\big(k-\gamma p(p-\lambda)\big)\right] b_k}{(\gamma p-\gamma\lambda-1)} \leq \frac{[(k-p)(\alpha(1+v)+1)-v(k-\beta p(p-\lambda))]^2 {b_k}^2}{2(\beta p-\beta\lambda-1)^2}, \quad (19)$$

$(p, n \in N)$

from (19), we have

$$\gamma \leq \frac{2vp(\beta p-\beta\lambda-1)^2[vk-(k-p)(\alpha(1+v)+1)] - \left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k}{(p-\lambda)\left[2v^2p^2(\beta p-\beta\lambda-1)^2 - \left[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\right]^2 b_k\right]}.$$

This completes the proofof Theorem 10.

6. Neighborhood propertyfor the Class $\Im S_g(p, v, \beta, \lambda)$

Now, following the earlier investigations by Goodman [7], Ruscheweyh[11], and others including Altintas and Owa [4], Altintas et al.([5] and [6]),Murugusundaramoorthy and Srivastava [8], Raina and Srivastava [10], Srivastava and Orhan [13] (see also [3] and [12]), we define theWe define the $(n, t)-$ neighborhood of a function $f(z) \in \Im_p(n)$ by

$$N_{n,t}(f) = \left\{ g \in \Im_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq t, 0 \leq t < 1 \right\}. \quad (20)$$

For the identity function $e(z) = z$, we have

$$N_{n,t}(e)=\left\{g\in\Im_p(n): g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}k|b_k|\le t, 0\le t<1\right\}. \tag{21}$$

Definition 2. A function $f\in\Sigma_p^*$ is said to be in the class $\Im S_g{}^{\omega}(p,v,\beta,\lambda)$if there exists a function $g\in\Im S_g(p,v,\beta,\lambda)$such that

$$\left|\frac{f(z)}{g(z)}-1\right|<p-\omega,\quad (z\in U, 0\le\omega<1).$$

Theorem 9. If $g\in\Im S_g(p,v,\beta,\lambda)$and

$$\omega=p-\frac{t\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]a_k}{\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]a_k-vp(\beta p-\beta\lambda-1)}. \tag{22}$$

Then $N_{n,t}(g)\subset\Im S_g{}^{\omega}(p,v,\beta,\lambda)$.

Proof. Let $f\in N_{n,\delta}(g)$. We want to find from (20) that

$$\sum_{k=n+p}^{\infty}k|a_k-b_k|\le t,$$

which readily implies the following coefficient inequality

$$\sum_{k=n+p}^{\infty}|a_k-b_k|\le t,\quad (n,p\in N). \tag{23}$$

And, since $g\in\Im S_g(p,v,\beta,\lambda)$, we have from Theorem 1

$$\sum_{k=n+p}^{\infty}b_k\le\frac{vp(\beta p-\beta\lambda-1)}{\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]a_k}.$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right|\le\frac{\sum_{k=n+p}^{\infty}|a_k-b_k|}{1-\sum_{k=n+p}^{\infty}b_k}$$

$$\le\frac{t\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]a_k}{\big[(k-p)(\alpha(1+v)+1)-v\big(k-\beta p(p-\lambda)\big)\big]a_k-vp(\beta p-\beta\lambda-1)}=p-\lambda.$$

Hence, by Definition (2), $f\in\Im S_g{}^{\omega}(p,v,\beta,\lambda)$for ω given by (22).
This completes the proof.

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