ON MKC-Spaces , MLC-Spaces and MH-Spaces

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Abstract:

The purpose of this paper is deal with the study of MKC-spaces(spaces whose minimal $KC-$spaces) and $MLC-spaces$ (spaces whose minimal $LC-spaces$)also we studied the relationships between minimal $KC-$spaces, minimal $LC-spaces$ and minimal Hausdroff spaces.

المستخلص

تهدف هذه الورقة لدراسة فضاءات $-KC$ الصغرى و فضاءات $-LC$ الصغرى كما درسنا في هذه الورقة العلاقات بين فضاءات $-KC$ الصغرى وفضاءات $-LC$ الصغرى وفضاءات هاوزدورف الصغرى.

KEYWORDS: $LC-space$, $P-space$, Lindelöf, $KC-space$.

1. Introduction:

Let P be a topological property, X be a nonempty and let $P(X)$ denote the set of all topologies on X having the property P. $P(X)$ is partially ordered by set inclusion. (X,T) is minimal P ($P-$minimal) if T is minimal in $P(X)$. In $[1]$ there is a good survey on minimal topologies and it stated there that every compact Hausdroff is minimal Hausdroff.

Radhi I.M. $[13]$ introduced a new concept of minimal $KC-$spaces and a new concept of minimal $LC-spaces$ .and showed that compact $KC-space$ is minimal $KC-space$ also showed that Lindelöf LC-space is minimal LC-space.

In this paper we show that Hausdorff minimal $KC-space$ is minimal Hausdorff, R_1 minimal $KC-space$ is minimal Hausdorff, regular minimal $KC-space$ is minimal Hausdorff, If X and Y are compact Hausdorff spaces then $X \times Y$ is minimal $KC-space$ and minimal Hausdorff space, If X and Y are compact Hausdorff $LC-spaces$, then $X \times Y$ is minimal $LC-space$ and minimal $KC-space$, If X and Y are Hausdorff Lindelof $LC-spaces$, then $X \times Y$ is minimal $LC-s\ p\ a$ and if X and Y are Hausdorff Lindelöf $P-spaces$ then $X \times Y$ is minimal $LC-space$ if and only if X and Y are minimal $LC-spaces$ .Our terminology is standard. The closure of a subset A of a space (X,T) is denoted by clA .The set of all positive integer is denoted by ω.

2. Minimal KC-spaces:

Definition2.1 [15]: A topological space (X,T) is a $KC-space$ if every compact subset of X is closed.

Definition2.2 [13]: Let (X,T) be a $KC-$space, we say that (X,T) is a minimal $KC-$space iff $T^* \subset T$ implies (X,T^*) is not a $KC-space$, (we will use MKC to denote minimal $KC-space$).

Definition2.3 [11]: Let (X,T) be a Hausdroff space, we say that (X,T) is a minimal Hausdroff space iff $T^* \subset T$ implies (X,T^*) is not a Hausdroff space, (we will use MH to denote minimal Hausdroff space).

Theorem2.4 [13]: Every compact $KC-$space is a MKC.

Corollary2.5: Every countably compact Lindelöf $KC-$space is a MKC.

Proof. Obvious by theorem 2.4

Theorem2.6 [13]: Every locally compact MKC is a MH.

Theorem2.7[13]: Every locally compact $KC-space$ X is a Hausdorff.

Theorem2.8[13]:

(i) Every Hausdorff space is a $KC-space$.

(ii) Every $KC-space$ is T_1.

Theorem2.9[13]: For a compact $KC-space$ X and $Y \subset X$ the following are equivalent:

(a) Y is a closed in X.

(b) Y is a compact in X.

Theorem2.10 [13]:

(i) The property of being $KC-space$ is a topological property.

(ii) The property of being $KC-space$ is a hereditary property.

Corollary2.11:

(i) Every closed subspace of compact $KC-space$ is a MKC .

(ii) Every subspace of hereditarily compact $KC-space$ is a MKC .

Proof . This is obvious by theorem 2.10and theorem 2.4.

Corollary2.12: For a compact $KC-space$ X and $Y \subset X$ the following are equivalent:

(a) Y is a closed in X .

(b) Y is a compact and $MKC-space$ in X .

Proof . (a) $\Rightarrow$(b): Let Y be a closed in X , then Y is a compact , so Y is a compact $KC-space$ by theorem2.10(ii),hence Y is a $MK-space$ by theorem2.4.

(b) $\Rightarrow$ (a) : Let Y be a compact $MK-space$ in X ,then Y is a compact $KC-space$, so Y is closed .

Theorem2.13: Every Hausdorff $MKC-space$ is a MH .

Proof . Let (X,T)be a Hausdorff $MKC-space$,then X is a Hausdorff $KC-space$.

Suppose X is not a $MH-space$,so there exists a topology T^* on X ,$T^* \subset T$ and (X,T^*) is a Hausdorff space implies that (X,T^*) is a $KC-space$ by theorem 2.8(i) which is a contradiction, therefore$(X,T)$is a MH .

Theorem2.14: Every compact $MH-space$ is a MKC .

Proof . Let X be a compact $MH-space$, then X is a compact Hausdorff, so X is a compact $KC-space$ by theorem2.8,hence X is a $MKC-space$ by theorem2.4.

Corollary2.15: For a compact Hausdorff space X the following are equivalent:

(a) X is a $MKC-space$.

(b) X is a $MH-space$.

Proof . This is obvious by theorem 2.13and theorem 2.14.

Definition2.16 [4]: A topological space (X,T) is a $R_1-space$ if x and y have disjoint neighborhoods whenever $cl\{x\} \neq cl\{y\}$ .Clearly a space is Hausdorff if and only if its T_1 and R_1 .

Theorem2.17: Every R_1 $MKC-space$ is a MH .

Proof . Let (X,T) be a R_1 $MKC-space$, then X is a R_1 $KC-space$,hence X is a Hausdorff by theorem2.8(ii) and definition 2.16.

Suppose X is not a $MH-space$, so there exists a topology T^* on X , $T^* \subset T$ and (X,T^*) is a Hausdorff space implies that (X,T^*) is a $KC-space$ by theorem 2.8(i)which is a contradiction, therefore (X,T) is a MH .

Theorem2.18: Every regular $MKC-space$ is a MH .

Proof . Let (X,T) be a regular $MKC-space$,then X is a regular $KC-space$,hence X is a Hausdorff by theorem2.8(ii).

Suppose X is not a $MH-space$,so there exists a topology T^* on X , $T^* \subset T$ and (X,T^*) is a Hausdorff space implies that (X,T^*) is a $KC-space$ by theorem 2.8(i) which is a contradiction, therefore (X,T) is a MH .

Theorem2.19 [13]:

Suppose $X \times Y$ is a compact $KC-space$, then each of X , Y is a $MKC-space$.

Corollary2.20:

Suppose $X \times Y$ is a regular compact $KC-space$, then each of X , Y is a $MH-space$.

Proof . Each of X and Y is a $MKC-space$ by theorem 2.19.Since X and Y are regular spaces, hence each of X , Y is a $MH-space$ by theorem 2.18.

Theorem2.21: If X and Y are compact Hausdorff spaces, then $X \times Y$ is a $MKC-space$ and a $MH-space$.

Proof . Since X and Y are compact Hausdorff spaces, then $X \times Y$ is a compact Hausdorff space, so $X \times Y$ is a compact $KC-space$ by theorem2.8 (i), hence $X \times Y$ is a $MKC-space$ by theorem2.4 and $X \times Y$ is a $MH-space$ by theorem2.13.

3. Minimal LC-spaces:

Definition3.1: A topological space (X,T) is an $LC-space$ if every Lindelöf subset of X is closed $[6],[10]$.Notice that $LC-space$ is also known under the name $L-closed$ $[5]$, $[7]$ and $[12]$.

Definition3.2 [13]: Let (X,T) be a $LC-space$ we say that X is a minimal $LC-space$ (MLC) iff $T^* \subset T$ implies (X,T^*) is not $LC-space$.

Definition3.3 [3]: A set F in a topological space is called $F_\sigma - closed$ if it is the union of at most countably many closed sets.

A set G is called a $G_\sigma - open$ if it is the intersection of at most countably many open sets.

Definition3.4[8]: A topological space (X,T) is called $P-space$ if every $G_\sigma - open$ set in X is open.

Theorem3.5 [13]: Every Lindelöf $LC-space$ is a MLC.

Theorem3.6 [6]: For a Hausdorff Lindelöf space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a $P-space$.

Corollary3.7: For a Hausdorff Lindelöf space X the following are equivalent:

(a) X is a $MLC-space$.

(b) X is a $P-space$.

Proof . This is obvious by theorem 3.5and theorem 3.6.

Corollary3.8: For a Hausdorff compact space X the following are equivalent:

(a) X is a $MLC-space$.

(b) X is a $P-space$.

Proof . This is obvious by theorem 3.5and theorem 3.6.

Theorem3.9 [8]: Every Huasdorff $P-space$ is an $LC-space$.

Corollary3.10: Every Hausdorff Lindelöf $P-space$ is a MLC.

Proof . This is obvious by theorem 3.9and theorem 3.5.

Theorem3.11[13]:

(i) Every $LC-space$ is a $KC-space$.

(ii) Every $LC-space$ is a $T_1-space$.

Theorem3.12: For a compact $P-space$ X the following are equivalent:

(a) X is a $MH-space$.

(b) X is a Hausdorff $MLC-space$.

Proof . (a) $\Rightarrow$(b): Let X be a $MH-space$, then X is a Hausdorff, so X is an $LC-space$ by theorem3.9,hence X is a $MLC-space$ by theorem3.5.

(b) $\Rightarrow$ (a): Let X be a Hausdorff $MLC-space$,then X is a Hausdorff $LC-space$, so X is a Hausdorff $KC-space$ by theorem3.11 (i).Since X is a compact, then X is a $MKC-space$ by theorem2.4, hence X is a $MH-space$ by theorem2.13.

Corollary3.13:

(i) Every compact $MLC-space$ is a MKC .

(ii) Every compact $LC-space$ is a MKC .

(iii) Every countably compact Lindelöf $LC-space$ is a MKC .

Proof . This is obvious by theorem 3.11(i) and theorem 2.4.

Theorem3.14 [13]:

(i) The property of being $LC-space$ is a topological property.

(ii) The property of being $LC-space$ is a hereditary property.

(iii)

Corollary3.15:

(i) Every closed subspace of compact $LC-space$ is a MLC and a MKC.

(ii) Every closed subspace of Lindelöf $LC-space$ is a MLC.

(iii) Every subspace of hereditarily Lindelöf $LC-space$ is a MLC.

Proof . This is obvious by theorem 3.14(ii), theorem 3.5, theorem3.11 (i) and theorem2.4.

Theorem3.16: For a compact $P-space$ X the following are equivalent:

(a) X is a $MKC-space$.

(b) X is a $MLC-space$.

Proof . (a) $\Rightarrow$(b): Let X be a $MKC-space$, then X is a $KC-space$, so X is a Hausdorff by theorem2.7.Since X is a $P-space$,then X is an $LC-space$ by theorem3.9,hence X is a $MLC-space$ by theorem 3.5.

(b) $\Rightarrow$ (a) : Let X be $MLC-space$,then X is an $LC-space$, so X is $KC-space$ by theorem3.11(i), hence X is a $MKC-space$ by theorem 2.4.

Definition3.17 [2]: A topological space (X,T) is called

(1) an $L_1-space$ if every Lindelöf $F_\sigma-closed$ is closed,
(2) an $L_2-space$ if clL is Lindelöf whenever $L \subseteq X$ is Lindelöf,
(3) an $L_3-space$s if every Lindelöf subset L is an $F_\sigma-closed$,
(4) an $L_4-space$ if whenever $L \subseteq X$ is Lindelöf, then there is a Lindelöf $F_\sigma-closed$ F with $L \subseteq F \subseteq clL$.

Theorem3.18 [2]:

(i) If (X,T) is an $LC-space$, then (X,T) is an $L_i-space$, i=1, 2, 3, 4.

(ii) If (X,T) is an $L_1-space$ and an $L_3-space$, then (X,T) is an $LC-space$.

(iii)Every $Q-set$ space is an $L_3-space$.

Definition3.19 [2]: A topological space (X,T) is called a $Q-set$ space if each subset of X is an $F_\sigma-$ closed sets.

Corollary3.20: Every $Q-set$ $L_1-space$ is an $LC-space$.

Proof. Let X be $Q-set$ space, then X is an $L_3-space$ by theorem 3.18(iii),since X is an $L_1-space$,then X is an $LC-space$ by theorem3.18(ii).

Theorem3.21: Every P $Q-set$ space X is an $LC-space$.

Proof. If L is a Lindelöf subset in X ,which is a $Q-set$ space, then L is an $F_\sigma-closed$ set, but X is a $P-space$, so L is a closed set , hence X is an $LC-space$.

Theorem3.22[3]: Countable union of Lindelöf subset is Lindelöf.

Theorem3.23: Every Lindelöf $L_1-space$ is a $P-space$.

Proof. For each $n\in\omega$, let A_n be closed in Lindelöf $L_1-space$ X and $A=\bigcup_{n\in\omega}A_n$, then A_n is a Lindelöf subset in X and thus A is a Lindelöf subset in X by theorem3.22. Since X is an $L_1-space$, then A is closed in X ,hence X is a $P-space$.

Theorem3.24: Every $PL_3-space$ is an $LC-space$..

Proof . Let L be a Lindelöf subset of X , then L is $F_\sigma-closed$ set (since X is an $L_3-space$),so L is closed set(since X is a $P-$ space) ,hence X is an $LC-space$.

Corollary3.25:

(i) Every Lindelöf $L_1L_3-space$ is a MLC.

(ii) Every Hausdorff Lindelof $L_1-space$ is a MLC.

(iii)Every Lindelof $Q-set$ $L_1-space$ is a MLC.

(iv)Every $LC-space$ having dense Lindelöf subset is a MLC.

(v) Every 2^{nd} countable (C_{11}) $LC-space$ is a MLC.

(vi)Every Lindelöf P $Q-set$ space is a MLC.

(vii) Every Lindelöf $PL_3 - space$ is a MLC.

(viii) Every compact $LC - space$ is a MLC.

Proof . Obvious

Corollary3.26:.

(i) Every compact Hausdorff space is a MKC.

(ii) Every compact $R_1T_1 - space$ is a MKC.

(iii) Every compact Tychonoff $P - space$ is a MKC.

(iv) Every compact $P \quad Q - set$ space is a MKC.

(v) Every compact $Q - set \; L_1 - space$ is a MKC.

(vi) Every compact $L_1L_3 - space$ is a MKC.

(vii) Every compact $L_1L_3 - space$ is a MKC.

(viii) Every compact $PL_3 - space$ is a MKC.

Proof . Obvious

Theorem3.27 [4]: If X and Y are Hausdorff $LC - spaces$, then $X \times Y$ is an $LC - space$.

Theorem3.28: If X and Y are compact Hausdorff $LC - spaces$, then $X \times Y$ is a $MLC - space$ and a $MKC - space$.

Proof . Since X and Y are Hausdorff $LC - spaces$, then $X \times Y$ is an $LC - space$ by theorem3.27, so $X \times Y$ is a compact $LC - space$,hence $X \times Y$ is a $MLC - space$ by corollary3.25(viii) and $X \times Y$ is a $MKC - space$ by corollary3.13(ii).

Theorem3.29 [6]: If X and Y are $LC - spaces$ and either X or Y is regular , then $X \times Y$ is an $LC - space$.

Theorem3.30: If X and Y are compact $LC - spaces$ and either X or Y is regular, then $X \times Y$ is a $MLC - space$ and a $MKC - space$.

Proof . Since X and Y are $LC-spaces$ and either X or Y is regular, then $X \times Y$ is an $LC-space$ by theorem3.29, so $X \times Y$ is a compact $LC-space$, hence $X \times Y$ is a $MLC-space$ by corollary3.25(viii) and $X \times Y$ is a $MKC-space$ by corollary3.13(ii).

Corollary3.31[4]: If X and Y are R_1 $LC-spaces$, then $X \times Y$ is an $LC-space$.

Theorem3.32: If X and Y are compact R_1 $LC-spaces$, then $X \times Y$ is a $MLC-space$ and a $MKC-space$.

Proof . Since X and Y are R_1 $LC-spaces$, then $X \times Y$ is an $LC-space$ by corollary3.31, so $X \times Y$ is a compact $LC-space$, hence $X \times Y$ is a $MLC-space$ by corollary3.25(viii) and $X \times Y$ is a $MKC-space$ by corollary3.13(ii).

Theorem3.33[14](Nobles Theorem):

If X and Y are Lindelöf $P-space$, then $X \times Y$ is a Lindelöf.

Theorem3.34[6]: Every Lindelof $LC-space$ is a $P-space$.

Theorem3.35:

If X and Y are Hausdorff Lindelöf $LC-spaces$, then $X \times Y$ is a $MLC-space$.

Proof . Since X and Y are Hausdorff $LC-spaces$, then $X \times Y$ is an $LC-space$ by theorem3.27.Since X and Y are Lindelöf $LC-spaces$, then X and Y are $P-space$ by theorem3.34, so $X \times Y$ is a Lindelöf by theorem3.33, hence $X \times Y$ is a $MLC-space$ by theorem3. 5.

Theorem3.36:

If X and Y are Lindelöf R_1 $LC-spaces$, then $X \times Y$ is a $MLC-space$.

Proof. Since X and Y are R_1 $LC-spaces$, then $X \times Y$ is an $LC-space$ by corollary3.31.Since X and Y are Lindelöf $LC-spaces$, then X and Y are $P-space$ by theorem3.34, so $X \times Y$ is a Lindelöf by theorem3.33, hence $X \times Y$ is a $MLC-space$ by theorem3. 5.

Theorem3.37:

If X and Y are Lindelöf $LC-spaces$ and either X or Y is regular, then $X \times Y$ is a $MLC-space$.

Proof. Since X and Y are $LC-spaces$ and either X or Y is regular, then $X \times Y$ is an $LC-space$ by theorem3.29, Since X and Y are Lindelöf $LC-spaces$,then X and Y are $P-space$ by theorem3.34, so $X \times Y$ is a Lindelöf by theorem3.33, hence $X \times Y$ is a $MLC-space$ by theorem3. 5.

Theorem3.38: If X and Y are Hausdorff Lindelöf $P-spaces$,

then $X \times Y$ is a $MLC-space$ if and only if X and Y are $MLC-spaces$.

Proof. Let $X \times Y$ be a $MLC-space$,then $X \times Y$ is an $LC-space$,so X and Y are $LC-spaces$ by theorem3.14(i)and(ii), hence X and Y are $MLC-spaces$ by theorem3.5.

Conversely, Let X and Y are $MLC-spaces$, then X and Y are $LC-spaces$,so $X \times Y$ is an $LC-space$ by theorem3.27. Since X and Y are Lindelöf $P-spaces$, then $X \times Y$ is a Lindelöf by theorem3.33, hence $X \times Y$ is a $MLC-space$ by theorem3.5.

Theorem3.39 [9]: Let X and Y be topological spaces. If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Theorem3.40 [9]: X is Hausdorff spaces if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.

Theorem3.41: Let X and Y be Hausdorff Lindelöf $LC-spaces$.

If A is closed in X and B is closed in Y, then $A \times B$ is a MLC.

Proof . Since X and Y are Hausdorff $LC-spaces$, then $X \times Y$ is an $LC-space$ by theorem3.27.Since X and Y are Lindelöf $LC-spaces$,then X and Y are $P-space$ by theorem3.34, so $X \times Y$ is a Lindelöf by theorem3.33, hence $A \times B$ is a MLC by theorem3. 39 and theorem3. 15(ii).

Theorem3.42: If X is a Hausdorff Lindelöf $LC-space$,then the diagonal $\Delta = \{(x,x) : x \in X\}$ is a MLC.

Proof. Since X is a Hausdorff $LC-spaces$, then $X \times X$ is an $LC-space$ by theorem3.27.Since X is a Lindelöf $LC-spaces$,then X is a $P-space$ by theorem3.34, so $X \times X$ is a Lindelöf by theorem3.33, hence the diagonal $\Delta = \{(x,x) : x \in X\}$ is a MLC by theorem 3. 40 and theorem 3. 15(ii).

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