# ON Locally Lindelöf Spaces,Co-Lindelöf Topologies and Locally LC-Spaces

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## Abstract:

The aim of this paper is to continue the study of locally Lindelöf spaces,Co- Lindelöf topologies and locally $LC-spaces$. We also study their relationships to $L_i - spaces \quad i = 1,2,3,4$ .

المستخلص

تهدف هذه الورقة دراسة فضاءات locally Lindelöf وتبولوجيات Co- Lindelöf و فضاءات $LC-$ locally وايضا درسنا علاقة علاقة هذه الفضاءات مع الفضاءات $L_i -$ $i = 1.2,3,4$.

**KEYWORDS:** $LC-space, P-space$, Lindelöf, locally Lindelöf spaces, Co- Lindelöf topologies, locally $LC-spaces$.

## 1. Introduction:

Dontchev, Ganster and Kanibir $[2]$introduced the class of locally Lindelöf and weakly locally Lindelöf by definitions, a topological space $(X,T)$ is called locally Lindelöf (resp. weakly locally Lindelöf) if each point of $X$ has a closed Lindelöf (resp. Lindelöf) neighborhood.

In 1984, Gauld, Mrsevic, Reilly and Vamanamurthy$[8]$ introduced the Co- Lindelöf topology of a given space$(X,T)$.They showed that $l(T)=\{\phi\}\cup\{G\in T : X-G \quad is \quad Lindelof \quad in \quad (X,T)\}$ is a topology on $X$ with $l(T)\subseteq T$, called the Co- Lindelöf topology of $(X,T)$.

Ganster, Kanibir and Reilly [6] introduced the class of locally $LC-spaces$ .By definition, a topological space $(X,T)$ is called a Locally $LC-space$ if each point of $X$ has a neighborhood which is an $LC-subspace$. In [6], the authors proved that a space $(X,T)$ is an $LC-space$ if each point of $X$ has a closed neighborhood that is an $LC-subspace$.Thus every regular locally $LC-space$ is an $LC-space$,a result first proved by Hdeib and Pareek in [10].

A set $F$ in a topological space is called $F_\sigma-closed$ if it is the union of at most countably many closed sets. A set $G$ is called a $G_\sigma-open$ if it is the intersection of at most countably many open sets [4].

In this paper, we consider and study of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$ .Furthermore, basic properties, preservation theorems and relationships of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$, are investigated. Moreover, to obtain several characterization and properties of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$ .

Our terminology is standard. The closure of a subset $A$ of a space $(X,T)$ is denoted by $clA$ .The set of all positive integer is denoted by $\omega$ .

## 2. locally Lindelöf and weakly locally Lindelöf :

**Definition2.1 [2]:** A topological space $(X,T)$ is called locally Lindelöf (resp. weakly locally Lindelöf) if each point of $X$ has a closed Lindelöf (resp. Lindelöf) neighborhood. It follows immediately from the definition that every locally Lindelöf space is a weakly locally Lindelöf.

Note that a weakly locally Lindelöf space need not be a locally Lindelöf space.

**Definition2.2:** A topological space $(X,T)$ is an $LC-space$ if every Lindelöf subset of $X$ is closed [7], [13].Notice that $LC-space$ is also known under the name $L-closed$ [9], [11] and [14].

**Definition2.3[12]:** A topological space $(X,T)$ is called $P-space$ if every $G_\sigma-open$ set in $X$ is open .

**Definition2.4 [2]:** A topological space $(X,T)$ is called

(1) an $L_1-space$ if every Lindelöf $F_\sigma-closed$ is closed,
(2) an $L_2-space$ if $clL$ is Lindelöf whenever $L\subseteq X$ is Lindelöf,
(3) an $L_3-space$s if every Lindelöf subset $L$ is an $F_\sigma-closed$ ,
(4) an $L_4-space$ if whenever $L\subseteq X$ is Lindelöf, then there is a Lindelöf

$F_\sigma-closed$ $F$ with $L\subseteq F\subseteq clL$ .

**Theorem2.5 [2]:**

(i) If $(X,T)$ is an $LC-space$, then $(X,T)$ is a $L_i-space$, i=1,2,3,4.

(ii) If $(X,T)$is an $L_1-space$ and an $L_3-space$, then$(X,T)$ is an $LC-space$.

(iii) ) Every space which is $L_1-space$ and $L_4-space$ is an $L_2-space$.

(iv) Every $L_2-space$ is an $L_4-space$ and every $L_3-space$ is an $L_4-space$.

(v) Every $L_3-space$ is $T_1$.

(vi) Every Lindelöf space is an $L_2-space$ ,and every $L_2-space$ having a dense Lindelöf Subset is Lindelöf.

(vii) Every $P-$ space is an $L_1-space$.

**Definition2.6 [2]:** A topological space $(X,T)$ is called a $Q-set$ space if each subset of $X$ is an $F_\sigma-closed$ sets.

**Theorem2.7 [12]:** Every Huasdorff $P-space$ is an $LC-space$.

**Corollary2.8:** Every Tychonoff $P-space$ is an $LC-space$.

**Proof.** Obvious.

**Proposition2.9 [2]:** Every weakly locally Lindelöf $L_2-space$ is locally Lindelöf,and so Every weakly locally Lindelöf space which is $L_1 and L_4$ is locally Lindelöf.

**Theorem2.10 [2]:**

Every locally Lindelöf space$(X,T)$ is an $L_1-space$ if and only if it is a $P-space$.

**Corollary2.11 [2]:** Every Huasdorff , locally Lindelöf $L_1-space$ is an $LC-space$.

**Corollary2.12 [2]:** Every weakly locally Lindelöf $LC-space$ $(X,T)$ is a $P-space$.

**Corollary 2.13:** For a Lindelöf space $X$ the following are equivalent:

(a) $X$ is locally Lindelöf.

(b) $X$ is a weakly locally Lindelöf.

**Proof.** This is obvious by definition 2.1.

**Corollary 2.14:** For an $L_2-space$ $X$ the following are equivalent:

(a) $X$ is locally Lindelöf.

(b) $X$ is a weakly locally Lindelöf .

**Proof.** (a) $\Rightarrow$(b): This is obvious by definition 2.1.

(b) $\Rightarrow$(a): This is obvious by proposition 2.9.

**Corollary 2.15:** For a $LC-space$ $X$ the following are equivalent:

(a) $X$ is locally Lindelöf .

(b) $X$ is a weakly locally Lindelöf .

**Proof.** (a) $\Rightarrow$(b): This is obvious by definition 2.1.

(b) $\Rightarrow$(a) : This is obvious by theorem 2.5(i)and proposition 2.9.

**Theorem 2.16:** For a Hausdorff locally Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is an $L_1-space$.

(c) $X$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).

(b) $\Rightarrow$(a): Let $X$ be an $L_1-space$, since $X$ is a Hausdorff locally Lindelof space, then $X$ is an $LC-space$ by corollary 2.11.

(b) $\Rightarrow$(c) : This is obvious by theorem 2.10.

(c) $\Rightarrow$(b) : This is obvious by theorem 2.10.

**Theorem2.17:** For Hausdorff weakly locally Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b):This is obvious by corollary 2.12.

(b) $\Rightarrow$(a):This is obvious by theorem 2.7.

**Theorem2.18:** Every $P$ $Q-set$ space $X$ is an $LC-space$ .

**Proof.** If $L$ is a Lindelöf subset in $X$ ,which is a $Q-set$ space, then $L$ is an $F_\sigma-closed$ set, but $X$ is a $P-space$, so $L$ is a closed set , hence $X$ is an $LC-space$ .

**<u>Theorem2.19:</u>** For a weakly locally Lindelöf $Q-set$ space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by corollary 2.12.

(b) $\Rightarrow$(a): This is obvious by theorem 2.18.

**<u>Theorem2.20:</u>** For a locally Lindelöf $Q-set$ space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$.

(c) $X$ is an $L_1-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i)and theorem2.10.

(b) $\Rightarrow$(a): This is obvious by theorem 2.18.

(b) $\Rightarrow$(c): This is obvious by theorem 2.10.

(c) $\Rightarrow$(b): This is obvious by theorem 2.10.

**<u>Corollary2.21:</u>** For a weakly locally Lindelöf $L_2-space$ $X$ the following are equivalent:

(a) $X$ is an $L_1-space$.

(b) $X$ is a $P-space$.

**Proof.** This is obvious by proposition 2.9 and theorem 2.10.

**<u>Theorem 2.22:</u>** For a locally Lindelöf $L_3-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is an $L_1-space$.

(c) $X$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).

(b) $\Rightarrow$(a): This is obvious by theorem 2.5(ii).

(b) $\Rightarrow$(c): This is obvious by theorem 2.10.

(c) $\Rightarrow$(b): This is obvious by theorem 2.10.

**Corollary2.23:**

(i) Every weakly locally Lindelöf $LC-space$ is locally Lindelof.

(ii) Every $LC-space$ having a dense Lindelöf Subset is locally Lindelof.

**Proof.** Obvious.

**Corollary2.24:** For a regular locally Lindelöf $L_1-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is $T_1$.

**Proof.** (a) $\Rightarrow$(b): Obvious.

(b) $\Rightarrow$(a): Let $X$ be a $T_1-space$,since $X$ is a regular, then $X$ is a Hausdorff .

Since $X$ is a locally Lindelöf $L_1-space$, then $X$ is an $LC-space$ by corollary 2.11.

**Definition2.25[5]:** A topological space $(X,T)$is a $R_1-space$ if $x$ $and$ $y$ have disjoint neighborhoods whenever $cl\{x\}\neq cl\{y\}$ .Clearly a space is Hausdorff if and only if its $T_1$ $and$ $R_1$.

**Corollary2.26:** For $R_1$ locally Lindelöf $L_1-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is $T_1$.

**Proof.** (a) $\Rightarrow$(b): Obvious.

(b) $\Rightarrow$(a): Let $X$ be a $T_1-space$,since $X$ isa $R_1$,then $X$ is a Hausdorff by definition2.25.

Since $X$ is a locally Lindelöf $L_1-space$,then $X$ is an $LC-space$ by corollary 2.11.

**Theorem2.27:**

For a Tychonoff weakly locally Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-$ space.

**Proof.** (a) $\Rightarrow$(b):This is obvious by corollary 2.12.

(b) $\Rightarrow$(a):This is obvious by corollary 2.8.

**Theorem2.28:** For $P-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is an $L_3-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).

(b) $\Rightarrow$ (a): Let $L$ be a Lindelöf subset of $X$ ,then $L$ is $F_\sigma-closed$ set (since $X$ is an $L_3-space$),so $L$ is closed set( since $X$ is a $P-space$) ,hence $X$ is an $LC-space$ .

**Theorem2.29:** For a weakly locally Lindelöf $L_3-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by corollary 2.12.

(b) $\Rightarrow$(a):This is obvious by theorem 2.28.

**Corollary2.30:** Every weakly locally Lindelöf $P$ $L_3-space$ is locally Lindelöf.

**Proof.** Obvious.

**Theorem2.31:** For a Hausdorff locally Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$ and an $L_2-space$.

**Proof.** (a) $\Rightarrow$(b): Let $X$ be an $LC-space$, then $X$ is an $L_2-space$ and an $L_1-space$ . Since $X$ is a locally Lindelöf, then $X$ is a $P-space$ by theorem 2.10.

(b) $\Rightarrow$(a):Let $L$ be a Lindelöf subset of $(X,T)$ and let $x \notin L$ .Since $(X,T)$ is Hausdorff, for each $y \in L$ there exist an open set $V_y$ containing $y$ with $x \notin clV_y$.

Clearly $\{V_y : y \in L\}$ is a cover of $L$ and so there exists a countable set $C \subseteq L$ such that $L \subseteq \bigcup_{y\in C} V_y \subseteq \bigcup_{y\in C} clV_y$ .For each $y \in C$ , $L \bigcap clV_y$ is Lindelöf and so $cl(L \bigcap clV_y)$ is Lindelöf since $(X,T)$ is an $L_2-space$.

Furthermore, if $W=\bigcup_{y\in C} cl(L\cap clV_y)$ then $W$ is a Lindelöf $F_\sigma-closed$ set and, since $(X,T)$ is a $P-space$, $W$ is a closed Lindelöf set not containing $x$. Thus $x\notin clL$. This shows that $L$ is closed in $(X,T)$.

**Theorem2.32:** Every $Q-set$ $L_1-space$ is an $L_2-space$.

**Proof.** Let $L$ be a Lindelöf subset of $X$, then $L$ is an $F_\sigma-closed$ set (since $X$ is a $Q-set$ space),so $L$ is closed set(since $X$ is an $L_1-space$),then $L=clL$ and $clL$ is a Lindelöf. Hence $X$ is an $L_2-space$.

**Theorem2.33:** For a locally Lindelöf $Q-set$ space $X$ the following are equivalent:

(a) $X$ is an $L_1-space$.

(b) $X$ is a $P-space$ and an $L_2-space$.

**Proof.** (a) $\Rightarrow$(b): Let $X$ be an $L_1-space$,since $X$ is a $Q-set$ space, then $X$ is an $L_2-space$ by theorem 2.32. Since $X$ is a locally Lindelöf $L_1-space$, then $X$ is a $P-space$ by theorem 2.10.

(b) $\Rightarrow$(a): This is obvious by theorem 2.5(vii).

**Definition2.34 [2]:** A topological space $(X,T)$ is called aweak $P-space$ if any countable union of regular closed sets is closed. One can show easily that $(X,T)$is aweak $P-space$ if and only if for every countable family $\{U_n : n\in\omega\}$of open sets, $cl\left(\bigcup_{n\in\omega} U_n\right)=\bigcup_{n\in\omega} clU_n$ .

**Corollary2.35:** Every $P-space$ $(X,T)$ is a weak $P-space$.

**Proof.** Let $F$ be a countable union of regular closed sets in $P-space$ $(X,T)$, then $F$ is an is an $F_\sigma-closed$ set, so $F$ is a closed set(since $X$ is a $P-space$ ),hence $X$ is a weak $P-space$.

**Corollary2.36:**

(i) Every locally Lindelöf $LC-space$ $(X,T)$ is weak $P-space$.

(ii) Every weakly locally Lindelöf $LC-space$ $(X,T)$ is weak $P-space$.

(iii) Every Lindelöf $LC-space$ $(X,T)$ is weak $P-space$.

**Proof.** Obvious.

**Corollary2.36:** For a Hausdorff locally Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a $P-space$ and an $L_2-space$.

(c) $X$ is an $L_1-space$.

**Proof.** This is obvious by theorem 2.16 and theorem 2.33.

**Theorem2.37:** Every locally Lindelöf weak $P-space$ $(X,T)$ is $L_4-space$.

**Proof.** Let $L$ be a Lindelöf subset of $(X,T)$. Each point of $L$ has an open neighborhood $U_x$ such that $clU_x$ is Lindelöf. Pick accountable subset $C$ of $L$ such that $L\subseteq\bigcup_{x\in C}U_x$ .Since $(X,T)$ is aweak $P-space$ we have $clL\subseteq\bigcup_{x\in C}clU_x=W$ .Since $W$ is Lindelöf we conclude that $clL$ is Lindelöf and closed ,so $clL$ is Lindelöf $F_\sigma-closed$ set ,hence $(X,T)$ is an $L_4-space$.

## 3. Co- Lindelöf Topologies:

**Theorem3.1 [2]:** For a $space$ $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $L_1-space$.

(b) $(X,l(T))$ is a $P-space$.

**Corollary3.2:** If $(X,T)$ is Lindelöf space then $l(T)=T$.

**Proof**: Obvious.

**Corollary3.3:** If $(X,T)$ is an $LC-space$ then $(X,l(T))$ is a $P-space$.

**Proof**: This is obvious by theorem 2.5(i) and theorem 3.1.

**Theorem3.4:** For a $L_3-space$ $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

**Proof.** (a) $\Rightarrow$ (b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$ ,then $(X,T)$ is an $L_1-space$ by theorem 3.1.

Since $(X,T)$ is an $L_3-space$ , then $(X,T)$ is an $LC-space$ by theorem 2.5(ii).

**Theorem3.5 [1]:** Every $Q-set$ space is an $L_3-space$ .

**Corollary3.6:** Every $Q-set$ $L_1-space$ is an $LC-space$ .

**Proof.** Let $X$ be $Q-set$ space, then $X$ is an $L_3-space$ by theorem3.5,since $X$ is an $L_1-space$ ,then $X$ is an $LC-space$ by theorem2.5(ii).

**Theorem3.7:** For a $Q-set$ space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $LC-space$ .

(b) $(X,l(T))$ is a $P-space$ .

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$ ,then $(X,T)$ is an $L_1-space$ by theorem 3.1.

Since $(X,T)$ is a $Q-set$ space, then $(X,T)$ is an $LC-space$ by corollary 3.6.

**Theorem3.8:** For a locally Lindelöf space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is a $P-space$ .

(b) $(X,l(T))$ is a $P-space$ .

**Proof.** (a) $\Rightarrow$(b): Let $(X,T)$ be a $P-space$ .Since $(X,T)$ is a locally Lindelof space,then $(X,T)$ is an $L_1-space$ by theorem 2.10,hence $(X,l(T))$ is a $P-space$ by theorem 3.1.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$ ,then $(X,T)$ is an $L_1-space$ by theorem 3.1.

Since $(X,T)$ is a locally Lindelöf space, then $(X,T)$ is an $P-space$ by theorem 2.10.

**Definition3.9 [1]:** A topological space $(X,T)$ is said to be anti – Lindelöf if each Lindelof subset of $X$ is countable.

**Theorem3.10 [2]:** Every $T_1$ anti- Lindelöf space is an $L_3-space$ .Hence every $T_1$, anti-Lindelöf $L_1-space$ is an $LC-space$ .

**Theorem3.11:** For a $T_1$ anti- Lindelöf space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1. Since $(X,T)$ is $T_1$ anti- Lindelöf space, then$(X,T)$is an $LC-space$ by theorem 3.10.

**Theorem3.12 [2]:** For a Hausdorff space $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is an $L_1-space$ and an $L_2-space$.

**Theorem3.13:** For a Hausdorff $L_2-space$ $(X,T)$the following are equivalent:

(a) $(X,T)$ is an $LC-space$.

(b) $(X,l(T))$is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since $(X,T)$ is a Hausdorff $L_2-space$, then $(X,T)$is an $LC-space$ by theorem 3.12.

**Theorem3.14 [4]:** Acountable union of Lindelöf subset is Lindelöf.

**Theorem3.15:** Every Lindelöf $L_1-space$ is a $P-space$.

**Proof.** For each $n \in \omega$, let $A_n$ be closed in Lindelöf $L_1-space$ $X$ and $A=\bigcup_{n\in\omega} A_n$ ,then $A_n$ is a Lindelöf subset in $X$ and thus $A$ is a Lindelöf subset in $X$ by theorem3.14. Since $X$ is an $L_1-space$, then $A$ is closed in $X$ ,hence $X$ is a $P-space$.

**Theorem3.16:** For a Hausdorff Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is an $L_1-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).

(b) $\Rightarrow$(a): Let $X$ be an $L_1-space$, since $X$ is a Lindelöf space ,then $X$ is a $P-space$ by theorem 3.15.Since $X$ is a Hausdorff space, then $X$ is an $LC-space$ by theorem 2.7.

**Theorem3.17:** For a Hausdorff Lindelöf space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

**Proof**: (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since $(X,T)$ is a Hausdorff Lindelöf space, then $(X,T)$ is an $LC-space$ by theorem3.16.

**Theorem3.18:** For a Hausdorff locally Lindelöf space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

**Proof**: (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ theorem 3.1.

Since $(X,T)$ is a Hausdorff locally Lindelöf space, then $(X,T)$ is an $LC-space$ by corollary 2.11.

**Corollary3.19:** If $(X,T)$ is an $L_1-space$ then $(X,l(T))$ is an $L_1-space$.

**Proof.** Let $(X,T)$ be an $L_1-space$, then $(X,l(T))$ is a $P-space$ by theorem3.1.Hence $(X,l(T))$ is an $L_1-space$ by theorem 2.5(vii).

**Definition3.20[7]:** A topological space $(X,T)$ is $cid-space$ if every countable subset of $X$ is closed and discrete.

**Remark3.21[7]:** Every $LC-space$ is $cid-space$.

**Theorem3.22:** For anti – Lindelöf space $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is $cid-space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by remark 3.21.

(b) $\Rightarrow$ (a): Let $L$ be a Lindelöf subset of $X$, then $L$ is countable ( since $X$ is anti – Lindelöf), so $L$ is a closed set(since $X$ is $cid-space$ ), hence $X$ is an $LC-space$.

**Corollary3.23:** If $(X,T)$ is a anti- Lindelöf $cid-space$ then $(X,l(T))$ is a $P-space$.

**Proof**: Let $(X,T)$ be a anti- Lindelöf $cid-space$, then $(X,T)$ is an $LC-space$ by theorem 3.22.Hence $(X,l(T))$ is a $P-space$ by corollary 3.3.

**Theorem3.24 [2]:** Every $T_1 L_1$ $-$ space is cid.

**Theorem3.25:** For a $T_1$ anti- Lindelöf space $(X,T)$ the following are equivalent:

(a) $(X,T)$ is $cid-space$.

(b) $(X,l(T))$ is a $P-space$.

**Proof.** (a) $\Rightarrow$(b): Let $(X,T)$ be $cid-space$.Since $(X,T)$ is a anti-Lindelof space, Then $(X,T)$ is an $L_1-space$ by theorem 3.22and theorem 2.5(i), hence $(X,l(T))$ is a $P-space$ by theorem 3.1.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$ is an $L_1-space$ by theorem3.1. Since $(X,T)$ is $T_1$ space, then $(X,T)$ is $cid-space$ by theorem 3.24.

**Theorem3.26:** If $(X,l(T))$ is a Lindelöf $LC-space$ then $(X,T)$ is an $L_1-space$.

**Proof.** For each $n\in\omega$, let $A_n$ be closed and Lindelöf in $(X,T)$ and let $A=\bigcup_{n\in\omega}A_n$. Since $(X,l(T))$ is a Lindelöf $LC-space$, then each $A_n$ is closed and Lindelöf in $(X,l(T))$ and so $A$ is also closed and Lindelöf in $(X,l(T))$ by theorem3.14. .Hence $A$ is closed in $(X,T)$ and so $(X,T)$ is an $L_1-space$.

## 4. Locally LC- spaces:

**Definition4.1 [6]:** A topological space $(X,T)$ is called a Locally $LC-space$ if each point of $X$ has a neighborhood which is an $LC-subspace$.

Clearly every $LC-space$ is locally $LC-space$ .In general the converse needs not be true $[5]$, however every regular locally $LC-space$ is $LC-space$ .

**Definition4.2[6]:** A topological space $(X,T)$ is called an $LC-space$ if each point of $X$ has a closed neighborhood that is an $LC-subspace$.

**Theorem4.3[3]:** Every locally $LC-space$ is $T_1$.

**Theorem4.4:** For a regular $P-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is a locally $LC-space$ .

(c) $X$ is an $T_1-space$ .

**Proof.** (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a): This is obvious by definition4.1.

(b) $\Rightarrow$ (c):This is obvious by theorem 4.3.

(c) $\Rightarrow$ (b):Let $X$ be a $T_1-space$,since $X$ is a regular ,then $X$ is a Hausdorff.

Since $X$ is a $P-space$, then $X$ is an $LC-space$ by theorem 2.7, hence $X$ is a locally $LC-space$ by definition4.1.

**Corollary4.5:** For a Finite topological $space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$.

(b) $X$ is a locally $LC-space$.

**Proof.** Obvious .

**Theorem4.6:** For a $R_1$ $P-space$ $X$ the following are equivalent:

(a) $X$ is an $LC-space$ .

(b) $X$ is a locally $LC-space$ .

(c) $X$ is $T_1-space$.

(d) $X$ is a Hausdorff space .

**Proof.** (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a): This is obvious by theorem 4.3, definition2.25and definition4.1.

(b) $\Rightarrow$(c): This is obvious by theorem 4.3.

(c) $\Rightarrow$ (b): Let $X$ be a $T_1 - space$,since $X$ is a $R_1$ ,then $X$ is a Hausdorff by definition 2.25.Since $X$ is a $P - space$, then $X$ is an $LC - space$ by theorem 2.7, hence $X$ is a locally $LC - space$ by definition4.1.

(c) $\Rightarrow$ (d):This is obvious by definition2.25.

(d) $\Rightarrow$ (c):Obvious .

**Theorem4.7:** For a regular $L_1$ $L_2 - space$ $X$ the following are equivalent:

(a) $X$ is a locally $LC - space$ .

(b) $X$ is $T_1 - space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 4.3.

(b) $\Rightarrow$ (a): Let $X$ be a $T_1 - space$,since $X$ is a regular $L_1$ $L_2 - space$ , then $X$ is an $LC - space$ by theorem 3.12, hence $X$ is a locally $LC - space$ by definition4.1.

**Theorem4.8:** For a regular locally Lindelöf $L_1 - space$ $X$ the following are equivalent:

(a) $X$ is a locally $LC - space$ .

(b) $X$ is $T_1 - space$.

**Proof.** (a) $\Rightarrow$(b): This is obvious by theorem 4.3.

(b) $\Rightarrow$ (a) : Let $X$ be a $T_1 - space$,since $X$ is a regular locally Lindelöf $L_1 - space$ , then $X$ is an $LC - space$ by corollary 2.11,hence $X$ is a locally $LC - space$ by definition4.1.

**Theorem4.9 [3]:** Every locally compact Hausdorff space is $T_3$ .

**Theorem4.10:** For a locally compact $R_1 - space$ $X$ the following are equivalent:

(a) $X$ is an $LC - space$ .

(b) $X$ is a locally $LC - space$ .

**Proof.** (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a) : Let $X$ be a locally $LC - space$,then $X$ is a $T_1 - space$ by theorem 4.3. Since $X$ is a $R_1 - space$ , then $X$ is a Hausdorff by definition2.25. Since $X$ is a locally compact, so $X$ is a regular by theorem 4.9 , hence $X$ is an $LC-space$ by definition 4.1.

**Proposition4.11 [3]:** For a $space\ X$ the following are equivalent:

(a) $X$ is a locally $LC-space$ .

(b) Every point of $X$ has an open neighborhood, which is an $LC-subspace$ of $X$ .

**Theorem4.12:** Every $2^{nd}$ countable$\left(C_{11}\right)$, a locally $LC-space$ is discrete.

**Proof.** Let$\left(X,T\right)$ be $2^{nd}$ countable and a locally $LC-space$ . We may assume that every point $x\in X$ has an open neighborhood $U$ that is both hereditarily Lindelöf and an $LC-space$ (since $X$ is a $2^{nd}$ countable and by Proposition 4.11).But this means that $U$ is an open discrete subspace of $\left(X,T\right)$.Hence $\left(X,T\right)$ is discrete.

**Theorem4.13:** If $\left(X,T\right)$ a regular space has an open cover by locally $LC-subspaces$ ,

then $X$ is an $LC-space$.

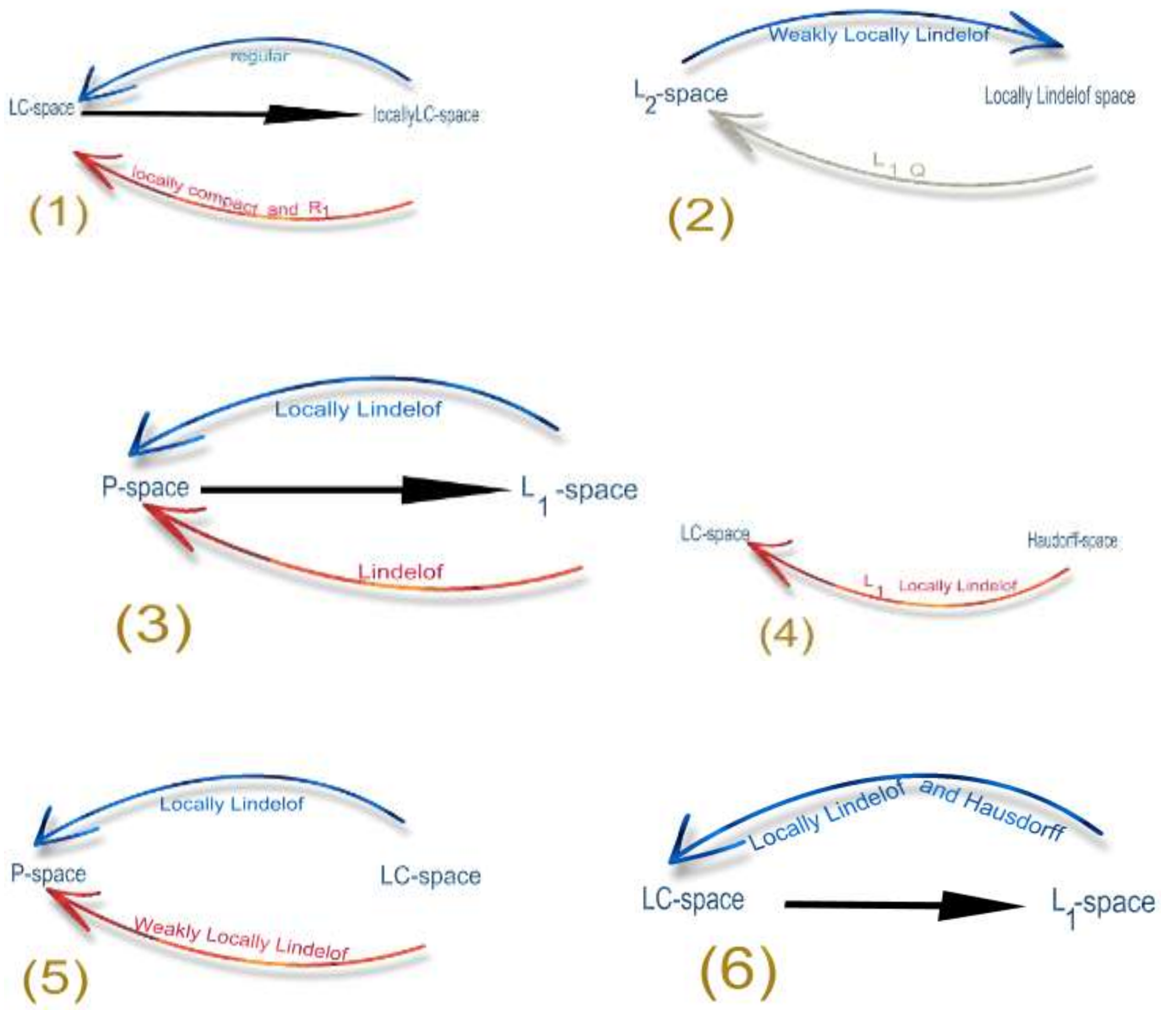
**Proof.** Let $X=\bigcup_{i\in I}G_i$ be an open cover of $X$ where each $G_i$ is a locally $LC-space$ ,and let $x\in X$ .Choose $j\in I$ such that $x\in G_j$ .If $U_J$ is an open and closed neighborhood(since $X$ is a regular)of $x$ in $G_j$ such that $U_J$ is an $LC-space$ of $G_j$ ,then $U_J$ is also open and closed in $\left(X,T\right)$. By definition4.2, $\left(X,T\right)$ is an $LC-space$.

**Theorem4.14:** If $\left(X,T\right)$ a regular space has an open cover by $LC-subspaces$ ,

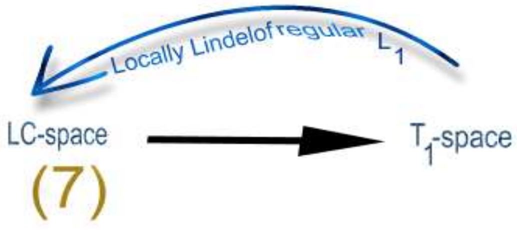
then $X$ is an $LC-space$.

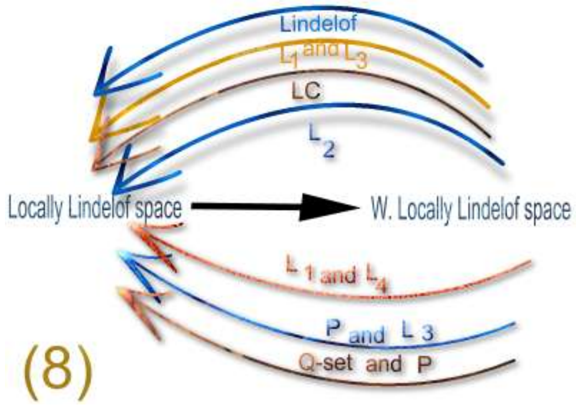
**Proof.** Let $X=\bigcup_{i\in I}G_i$ be an open cover of $X$ where each $G_i$ is $LC-space$ ,and let $x\in X$ .Choose $j\in I$ such that $x\in G_j$ .If $U_J$ is a closed neighborhood(since $X$ is a regular)of $x$ in $G_j$ such that $U_J$ is an $LC-space$ of $G_j$ ,then $U_J$ is also closed in $\left(X,T\right)$. By definition4.2, $\left(X,T\right)$ is an $LC-space$.

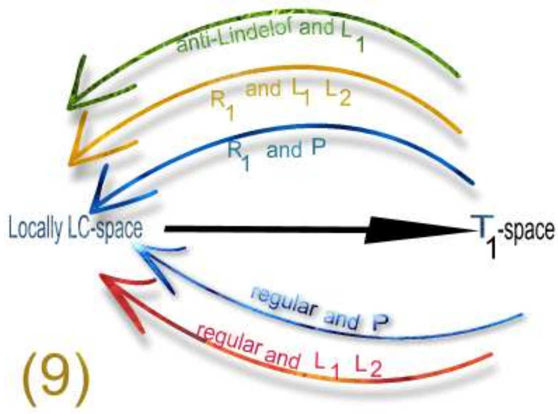
We have the following diagrams:



regular
LC-space
locallyLC-space
locally compact and R1
(1)
Weakly Locally Lindelof
L2-space
Locally Lindelof space
L1 Q
(2)
Locally Lindelof
P-space
L 1 -space
Lindelof
(3)
LC-space
Haudorff-space
L1 Locally Lindelof
(4)
Locally Lindelof
P-space
LC-space
Weakly Locally Lindelof
(5)
Locally Lindelof and Hausdorff
LC-space
L1-space
(6)




# Refereces


[1]Bankston P., The total negation of a topological property , Illinois J. Math. 23 (1979), 241-255.

[2] Dontchev J., Ganster M. and Kanibir A., On Some Generalization of LC- spaces,Acta Math. Univ. Comenianae Vol. LXVIII, 2(1999), pp. 345-353.

[3] Dontchev J., Ganster M. and Kanibir A., On locally LC- spaces, Math.GN. Vol. arXiv,7(1998), pp. 1-5.

[4] Engelking R. ,General Topology. Heldermann Verlag, Berlin, revised and completed edition,1989.

[5] Dontchev J., Ganster M., On the product of $LC-spaces$ , Q & A in General Topology15 (1997), 71–74 .

[6] Ganster M.,Kanibir A.,Reilly I., Two Comments Concerning Certain Topological Spaces ,Indian J., Pure Appl. Math.29(9)965-967,September1998..

[7] Ganster M. and Jankovic D., On spaces whose Lindelöf subsets are closed, Q & A in General Topology 7 (1989), 141–148

[8] Gauld D. B., Mrsevic M., Reilly I. and Vamanamurthy M. K., Co-Lindelof topologies and I-continuous functions, Glasnik Mat. 19(39) (1984),297-308.

[9] Hdeib H. Z., A note on L-closed spaces, Q & A in General Topology 6 (1988), 67-72.

[10] Hdeib H. Z. and Pareek C. M., On spaces in which Lindelof sets are closed, Q & A in General Topology 4 (1986), 3-13.

[11] Levy R., A non-P L-closed space, Q & A in General Topology 4 (1986), 145-146.

[12]Misra A. K. , A topological view of $P-spaces$ ,Topology & Appl.2(1972),349-362.

[13] Mukherji T. K. and Sarkar M., On a class of almost discrete spaces, Mat. Vesnik 3(16)(31)(1979), 459–474.

[14]Ori R. G., A note on L-closed spaces, Q & A in General Topology 4 (1986/87), 141– 143.