Shilkret integral based on binary-element sets and its application in the area of synthetic evaluation

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Abstract

The aim of this paper is to present a new framework for studying fuzzy measure and Shilkret integral which is alternative framework from usual fuzzy measure and Shilkret integral. First, we define a fuzzy measure based on binary-element sets, then, we propose Shilkret integral with respect to fuzzy measure based on binary-element sets, and we illustrate our framework by a practical example on synthetic evaluation.

Keywords: Binary-element sets, Fuzzy measures based on binary-element sets, Shilkret integral based on binary-element sets.

تكامل شالكريت مستند على مجموعات العنصر الثنائي وتطبيقة في حقل التقييم الصناعي

الخلاصة

الهدف من هذا البحث هو تقديم اطار جديد لدراسة القياس الضبابي وتكامل شالكريت ، فهو اطار بديل للقياس الضبابي الاعتيادي وتكامل شالكريت الاعتيادي. نَعرف اولا قياس ضبابي مستند على مجموعات العنصر الثنائي ثم نَقترح تكامل شالكريت متغلق بالقياس الضبابي المستند على مجموعات العنصر الثنائي ونشرح اطارنا المقترح بمثال تطبيقي حول التقييم الصناعي.

INTRODUCTION

The classical measure and integral theory is based on the additivity of set functions. Fuzzy measures [15] (capacities [3] or non-additive measures [4]) are obtained by replacing the additive requirement of classical measures with weaker requirement of monotonicity (with respect to set inclusion). Based on the fuzzy measure, several non-additive integrals (the Choquet integral [3], the Shilkret integral [13], the Sugeno integral [14], etc.) have been introduced in the last 60 years. In recent years there has been increasing interest in the development of new integrals useful in decision analysis or modeling of engineering problems [1,2,5,6,9,10,11,12].

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In this paper, we first propose a new framework for studying fuzzy measure and Shilkret integral through introducing the notion of binary-element sets. Secondly, we illustrate our framework by a practical example on multiple criteria decision analysis in the area of synthetic evaluation of objects in terms of multiple quality factors.

The structure of the paper is as follows. In the next section we recall basic definitions of usual fuzzy measures and Shilkret integral. In Sections 3 and 4, we present our alternative framework ``fuzzy measures based on binary-element sets'' and ``Shilkret integral based on binary-element sets''. In section 5, we illustrate our framework by a practical example on global evaluation of TV sets.

Throughout the paper, we denote R as the set of all real numbers, R+ the set of all non-negative real numbers, the universal set $X = \{1, \dots, n\}$ denotes a finite set of n elements (states of nature, criteria, individuals, etc), and $2^X = \{A \mid A \subset X\}$.

Usual fuzzy measures and Shilkret integral

In this section we recall basic definitions of usual fuzzy measures and Shilkret integral. Usual fuzzy measures on some finite universe are special monotone set functions defined in the following way (see, e.g. [8]).

Definition 1: A fuzzy measue on X is a set function $\mu : 2^X \to [0,1]$ satisfying the following requirements

(i) $\mu(\phi) = 0$ and $\mu(X) = 1$.

(ii) $\forall A, B \in 2^X$, $A \subset B$ implies $\mu(A) \le \mu(B)$.

The Shilkret integral is intimately related to the concept of fuzzy measures. Its definition in the case of a finite universal set is as follows (see, e.g. [13]).

Definition 2: Let μ be a fuzzy measure on X. The Shilkret integral of a vector $\boldsymbol{x} = (x_1, \cdots, x_i, \cdots, x_n)$; $x_i \in [0,1]$ with respect to μ, which is denoted by $(Sh)\int \boldsymbol{x}\, d\mu$ defined as

$$(Sh)\int \boldsymbol{x}\, d\mu = \vee_i^n [\sigma(x_i)\,.\, \mu(\{\sigma(1), \dots, \sigma(i)\})] \qquad \dots (1)$$

Where

$\vee$ is the maximum and σ is a permutation on X such that $x(\sigma(1)) \ge \cdots \ge x(\sigma(n))$.

Fuzzy measures based on binary-element sets

Bi-element sets

In the context of multi-criteria decision making problem, we shall consider a set $X = \{1, \dots, n\}$ of criteria and a set of alternatives described according to these criteria. We further assume that to each alternative is described by a vector

$\boldsymbol{x} = (x_1, \cdots, x_i, \cdots, x_n)$, $x_i \in [0,1]$ (or R+) and x_i expresses to which degree the alternative satisfies criterion $i \in X$. The problem is now to evaluate the alternatives with respect to criteria. Since the criteria have not always the same importance, and defining a fuzzy measure on a set of criteria can be seen as a way of modeling the interaction phenomena existing among these criteria. Hence, for every alternative, any element $i \in X$ has either positive effect (i.e., i is positively important criterion of weighted evaluation not only alone but also is interactive with other) or has negative effect (i.e., i is negatively important criterion). Thus, for every alternative, we can represent the element i as i^+ whenever i is positively important, and as i^+ whenever i is negatively important, and we call this element a binary-element (or simply bi-element). For this situation, we can describe the set of all possible combinations of binary elements (i.e. the possible outcomes of the alternative) of n criteria given by

$B(X) = \{ \ \{\tau_1, \cdots, \tau_n\} | \ \forall \, \tau_i \in \{i^+, i^-\}, \ i = 1,2 \cdots, n\}$ which is corresponds to power set $p(X)$ in the notation of classical set theory.

The set of all bi-element sets $B(X)$ can be identified with $\{0,1\}^2$, hence $|B(X)| = 2^n$. Also, any bi-element set $A \in B(X)$ can be written as a binary alternative denoted by $(\tau_1, \cdots, \tau_n)$ with $\tau_i = 1 \ if \ i^+ \in A \ and \ \tau_i = 0 \ if \ i^- \in A, \ i = 1,2 \cdots, n$.

We introduce the cardinality, complement, inclusion relation ⊆, union, and intersection of bi-element sets of $B(X)$as follows.

Definition 3: The cardinality of the bi-element set $A \in B(X)$ is number of important elements i^+ in A, denoted by a^+. That is, $a^+ = |A| = \sum_{i=1}^{n} \mathcal{X}_A(i^+)$ where

$$\mathcal{X}_A(i^+) = \begin{cases} 1 & if \ \ i^+ \in A \\ 0 & if \ \ i^+ \notin A \end{cases}$$

Definition 4: For any bi-element set $A \in B(X)$, we define the complement of A by $A^c = \left\{\{\tau_1{}^c, \ldots, \tau_i{}^c, \ldots, \tau_n{}^c\} \,\middle|\, \forall \, \tau_i \in A \right\}$ with $(i^+)^c = i^-$, $(i^-)^c = i^+$.

Definition 5: Let $A \in B(X)$, be the set of all bi-element sets and $A, \ B \in B(X)$.

Then, $A \subseteq B \ iff \ \forall \ i \in X$

$$if \ i^+ \in A \ \ implies \ i^+ \in B. \qquad \ldots (2)$$

Definition 6: Let $A \in B(X)$, be the set of all bi-element sets and $A, \ B \in B(X)$.

The union, $A \cup B$ of A and B is defined by,

$A \cup B = \left\{\tau_j \vee \tau_k | \ \tau_j \in A, \ \ \tau_k \in B, \ j = 1,2 \cdots, n, \ \ k = 1,2 \cdots, n \right\}$, with

$i^+ \vee i^- = i^+,\ i^+ \vee i^+ = i^+, i^- \vee i^- = i^-,\ i = 1,2 \cdots, n.$

Definition 7: Let $A \in B(X)$, be the set of all bi-element sets and $A,\ B \in B(X)$.
The union, $A \cap B$ of A and B is defined by,
$A \cap B = \{\tau_j \wedge \tau_k | \tau_j \in A,\ \tau_k \in B,\ j = 1,2 \cdots, n,\ k = 1,2 \cdots, n\}$, with
$i^+ \wedge i^- = i^-,\ i^+ \wedge i^+ = i^+, i^- \wedge i^- = i^-,\ i = 1,2 \cdots, n.$

Fuzzy measures based on bi-element sets

Based on the notion of bi-element sets in the above subsection, the following definition gives an equivalent definition of usual fuzzy measures.

Definition 8: Let $B(X)$ be the set of all bi-element sets. A set function $\mu: B(X) \to [0,1]$ is called capacity if it satisfies the following requirements:
(i) $\mu(X^-) = \mu(\{1^-, 2^- \cdots, n^-\}) = 0$ *and* $\mu(X^+) = \mu(\{1^+, 2^+ \cdots, n^+\}) = 1.$
(ii) $\forall\ A, B \in B(X)$, $A \subseteq B$ *implies* $\mu(A) \leq \mu(B).$
Furthermore,

- A fuzzy measure based on bi-element sets is said to be additive if it satisfies:
  $\mu(A \cup B) = \mu(A) + \mu(B).$
  for all $A,\ B \in B(X)$ with $A \cap B = X^-.$
- The conjugate (or dual) of a fuzzy measure based on bi-element sets μ is a fuzzy measure $\bar{\mu}$ defined by
$\bar{\mu}(A) = 1 - \mu(A^c),\ \ A \in B(X).$

The Shilkret integral based on the bi-element sets

In this section, we propose an equivalent model of Shilkret integral with respect to fuzzy measures based on the bi-element sets of real input $\boldsymbol{x}$. For an input vector $\boldsymbol{x} = (x_1, \cdots, x_i, \cdots, x_n)$ and $x_i \in [0,1]$ with $i \in \{1,2 \cdots, n\}$, and we consider a bi-element set $X^+ = \{i^+ | x_{i^+} \in R$ for each $i \in \{1,2 \cdots, n\}\}$, and each input value of x_i is expressed by x_{i^+} (to which degree the alternative satisfies positively important element i^+). Thus, we define the Shilkret integral with respect to fuzzy measure of real input x as follows.

Definition 9 Let $B(X)$ be the set of all bi-element sets and μ: B(X) → [0; 1] be a fuzzy measure based on bi-element sets. Then Shilkret integral of $\boldsymbol{x}$ with respect to μ is given by

$$\mathrm{Sh}_\mu(\boldsymbol{x}) = (Sh)\int \boldsymbol{x}\, d\mu = \bigvee_i^n \left[x_{\sigma(i^+)} \,.\, \mu\left(\{A_{\sigma(i^+)}\}\right)\right] \quad \ldots (3)$$

Where
$A_{\sigma(i^+)} = \{\ldots, \sigma((i-2)^-), \sigma((i-1)^-), \sigma(i^+), \ldots, \sigma(n^+)\}$ is bi-element set $\subseteq X^+$, and σ is a permutation on X^+ so that $x_{\sigma(i^+)} \leq \cdots \leq x_{\sigma(n^+)}.$

From the definition of the Shilkret integral with respect to the fuzzy measure based on bi-element sets we immediately see the following property.

Proposition 1 Let $B(X)$ be the set of all bi-element sets and μ: B(X) →[0; 1] be a fuzzy measure based on bi-element sets. Then Shilkret integral of $\boldsymbol{x}$ with respect to μ is given by

$$\mathrm{Sh}_{\mu}(\boldsymbol{x}) = (Sh)\int \boldsymbol{x}\, d\mu = \bigvee_{i}^{n} \left[x_{\sigma(i^+)} \cdot \mu\left(\left\{A_{\sigma(i^+)}\right\}\right)\right] \quad \text{... (4)}$$

Where

$A_{\sigma(i^+)} = \{\sigma(1^+), \dots, \sigma(i^+) \dots, \sigma((i+1)^-), \sigma((i+2)^-) \dots\}$ is bi-element set $\subseteq X^+$, and σ is a permutation on X^+ so that $x_{\sigma(i^+)} \geq \cdots \geq x_{\sigma(n^+)}$. ■

The next property shows the Shilkret integral with respect to the fuzzy measure based on bi-element sets of binary alternatives $(\tau_1, \cdots, \tau_n)$ equals fuzzy measure $\mu(A)$, $\forall\, A \in B(X)$. as shown by the following result.

Proposition 2 For any fuzzy measure based on bi-element sets (μ) on $B(X)$, the Shilkret integral of binary alternative $(\tau_1, \cdots, \tau_n)$ coincide with $\mu(A)$:

$$\mathrm{Sh}_{\mu}\big((\tau_1, \cdots, \tau_n)\big) = \mu(A), \qquad \forall\, A \in B(X).$$

Proof:

The Shilkret integral with respect to the fuzzy measure based on bi-element sets of binary alternative $(\tau_1, \cdots, \tau_n)$, that is of actions $\boldsymbol{x}$ such that $\tau_i = 1$ if $i^+ \in A$ and $\tau_i = 0$ if $i^- \in A$, $i = 1,2 \cdots, n$.

Then $x_{i^+} = 1$ for all $i \in A$, and $x_{i^-} = 1$ otherwise.

Thus

$$x_{1^+} = \cdots = x_{n^+} = 1$$

and

$$\mu\big(\{A_{\sigma(i^+)}\}\big) = \mu(A).$$

This gives (Formula (3))

$$\mathrm{Sh}_{\mu}\big((\tau_1, \cdots, \tau_n)\big) = \mu(A),$$
$$\forall\, A \in B(X).$$

Illustrative Example (Practical Example

Description of the example

Let us try to illustrate our approach by the example of evaluation of three TV sets adopted from [15]. We consider (for the sake of simplicity) only two quality factors : ``Picture'' and ``Sound''. These are denoted by P and S, respectively, and the corresponding weights are $w1 = 0.6$ and $w2 = 0.4$.

As an illustration, we give in Table 1, the scores for each factor and each TV set:

TV Set No.	Pacture (P)	Sound (S)
1	1	0
2	0	1
3	0.5	0.5

Table(1)

In the sequel, we will illustrate the two different approaches by applying them to above scores for global evaluation of three TV sets.

Solution attempt using the method of weighted mean

Computing the average evaluation of the three TV sets by using the method of weighted mean, we obtain the these synthetic evaluations of the three TV sets:

$$EW1 = w1 \times 1 + w2 \times 0 = 0.6$$
$$EW2 = w1 \times 0 + w2 \times 1 = 0.4$$
$$EW3 = w1 \times 0.5 + w2 \times 0.5 = 0.5.$$

From the above results. It can be seen that the first TV set is the best. Such a result is hardly acceptable since it does not agree with our intuition:

A TV set without any sound is not practical at all, even though it has an excellent picture. It is significant to realize that the cause of this counterintuitive result is not an improper choice of the weights. For example, if we chose

$w1 = 0.2$ and $w2 = 0.8$, we would have obtained

$$EW1 = w1 \times 1 + w2 \times 0 = 0.2$$
$$EW2 = w1 \times 0 + w2 \times 1 = 0.8$$
$$EW3 = w1 \times 0.5 + w2 \times 0.5 = 0.5.$$

Now, according to these results, the second TV set is identified as the best one, which is also counterintuitive: A TV set with good sound but no picture is not a real TV set, but just a radio.

We may conclude that, according to our intuition, the third TV set be identified as the best one: among the three TV sets, only the third one is really practical, even though neither picture nor sound are perfect. Thus, when using the method of weighted mean, no choice of the weights would lead to this expected result under the given scores.

Our approach

The method of weighted mean is based on an implicit assumption that the factors $x_1, x_2 \cdots, x_n$ are ``independent'' of one another. That is, their effects are viewed as additive. This, however, is not justifiable in some real problems. In this example, the importance of the combination of picture and sound is much higher then the sum of importance associated with picture and sound alone. If we adopt our approach

(a fuzzy measure based on binary-element sets) to characterize the importance of the two factors and, relevantly, use Shilkret integral based on binary-element sets as a synthetic evaluator of the quality of the three TV sets, a satisfactory result may be obtained. We consider $X = \{P, S\}$, the alternative consists of observing the factors where a TV has qualified. The possible outcomes of the alternative are the set of all possible combinations of binary elements of criteria (P, S). That is,
$B(X) = \{\{P^+, S^-\}, \{P^-, S^+\}, \{P^+, S^+\}, \{P^-, S^-\}\}$. For instance, the meaning of the bi-element set $\{P^+, S^-\}$ is as follows: the set of factors in which picture is important while sound is unimportant.

We interpret a value $\mu(A)$ of a fuzzy measure, as the ``degrees of importance'' of bi-element set $A \in B(X)$.

For instance, given the importance measure $\mu(\{P^+, S^-\}) = 0.4$, $\mu(\{P^-, S^+\}) = 0.3$, $\mu(\{P^+, S^+\}) = 1, \mu(\{P^-, S^-\}) = 0$, we obtain the following synthetic evaluation and by using Shilkret integral based on bi-element sets of each TV:

$ESh1 =$

$$(Sh)\int \boldsymbol{x_1}\, d\mu = (1\,.\,0.4) \vee (0.1) = 0.4$$

$ESh2 =$

$$(Sh)\int \boldsymbol{x_2}\, d\mu = (1\,.\,0.3) \vee (0.1) = 0.3$$

$ESh3 =$

$$(Sh)\int \boldsymbol{x_3}\, d\mu = (0.5\,.\,0) \vee (0.5.1) = 0.5$$

Where, x_1, x_2 and x_3 characterize the scores given for the three TV sets:
$x_1 = (1,0)$, $x_2 = (0,1)$, and $x_3 = (0.5.,0.5)$. Hence, we get a reasonable conclusion ``the third TV set is the best'' which agrees with our intuition. Consequently, the result is satisfactory.

Conclusions

In this paper, we have proposed alternative framework of fuzzy measure and Shilkret integral with respect to fuzzy measure. First, we have presented a fuzzy measure through attaching each element of the domain on the subsets of which the set function is defined. Secondly, we have proposed a new model for study Shilkret integral based on the notion of binary-element sets.

Finally, practical example is given for clarifying the usefulness of the proposed framework.

References:

[1] Abbas J., `` Logical twofold integral'', Engineering & Technology Journal, Vol. 28, No. 3; 2010.

[2] Bilbao, J.M., Fernandez, J.R., Jimenez Losada, A. and Lebron, E., ``Bi-cooperative Games'', First World Congress of the Game Theory Society (Games 2000) July 24-28, Bilbao, Spain, 2000.
[3] Choquet G., `` Theory of capacities'', Ann. Inst. Fourier, Vol. 5, pp. 131-295, 1953.
[4] Denneberg D. , `` Non-additive measure and integral'', Kluwer Academic Publisher, 1994.
[5] Felsenthal D. and Machover M., ``Ternary Voting games'' , International Journal of Game Theory, 26, pp. 335-351, 1997.
[6] Figueira J., Greco S., Ehrgott M.(Eds.), ``Multiple Criteria Decision Analysis: State of the Art Surveys'', Springer, Boston, 2005.
[7] Grabisch M., ``The symmetric Sugeno integral'', Fuzzy Sets Syst. 139, 3, pp. 473-490, 2003.
[8] Grabisch, M., Murofushi T. and Sugeno M., ``Fuzzy Measures and Integrals'', Physic-Verlag, 2000.
[9] Greco S. , Matarazzo B. , Slowinski R. , ``Bipolar Sugeno and Choquet integrals'', Proc. EUROFUSE Workshop on Information Systems, Varenna, Italy, pp. 191-196, 2002.
[10] Labreuche Ch. and Grabsch M., `` Generalized Choquet like-aggregation for handling ratio scales'', European J. Oper. Res.,172, pp. 931-955, 2006.
[11] Murofushi T. and Sugeno M., `` An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure'',
Fuzzy Sets and Systems, Vol. 29, pp. 201-227, 1989.
[12] Mesiar R., Mesiarov-Zemnkov A., Ahmad K., ``Discrete Choquet integral and some of its symmetric extensions'', Fuzzy Sets Syst, 184, pp. 148 -155, 2011.
[13] Shilkret N., ``Maxitive measure and integration'', Indag. Math. 74, pp. 109-116, 1971.
[14] Sugeno M., ``Theory of fuzzy integrals and its applications'', Ph.D. Thesis, Tokyo Institute of Technology, 1974.
[15] Wang Z. and Klir G.J., ``Fuzzy measure theory'', Plenum Press, New York, 1992.