A study of a prey-predator system with disease in prey

Dr. Raid Kamel Naji
College of science, University Baghdad/ Baghdad.
Dr. Basim N.Abood
Applied Science Department, University of Technology/ Baghdad.
Ala'a Abbas M.Rasheed
Applied Science Department, University of Technology/ Baghdad.
Email:P.emo15@yahoo.com

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Abstract

In this paper, the dynamical behavior of some eco-epidemiological models is investigated. Two types of prey-predator models involving infectious disease in prey population, which divided it into two compartments; namely susceptible population *S* and infected population *I*, are proposed and analyzed. The proposed model deals with SIS infectious disease that transmitted directly from external sources, as well as, through direct contact between susceptible and infected individuals. The model are represented mathematically by the set of nonlinear differential equations .The existence, uniqueness and boundedness of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. Finally, using numerical simulations to study the global dynamics of the model .

Keyword: prey-predator model; local stability; global stability.

دراسة نظام الفريسة المصابة /المفترس

الخلاصة

في هذه الرسالة ,تم بحث السلوك الديناميكي لبعض النماذج الوبائية.نوعان من نموذج الفريسة-المفترس تتضمن مرض معدي في مجتمع الفريسة ,والذي يقسم افراد مجتمع الفريسة الى قسمين رئسيين الافراد السليمة و الافراد المصابة اقترحت وحلت أن النموذج المقترح يتعامل مع مرض معدي والذي ينتقل من الافراد السليمة الى الافراد المصابة عن طريق المصدرالخارجية والا تصا ل المباشر.

تم تمثيل النوذج رياضيا بمجموعة من المعادلات التفاضلية الاعتيادية غير الخطية . وجود, وحدانية الحل وقيود الحل للنموذج بحثت وشروط الاستقرارية المحلية و الشاملة لكل نقاط التوازن الممكنة وضعت . واخيرا قمنا با لمحاكة العددية لدراسة ا لديناميكية الشاملة في النموذج .

INTRODUCTION:

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

Although the dynamics of two species prey-predator model with Holling type-II functional response received a lot of attention in literature, it is well known that in nature there are different factors in any given environment, such as disease, refuge, switches, age structure, etc. effect the dynamics of such model. Anderson and May [1] were the first who formulated a prey–predator model involving disease in prey species. Later on many researchers, especially in the last two decades, have proposed and studied different types of prey-predator models in the presence of disease in one of the species, see for example Haque and Chattopadhyay [2] which studied the role of transmissible diseases in a prey dependent prey-predator system with prey infection; Li et al [3] proposed the *SIS* model with a limited resource for treatment. In most of the previous studies, the only way of transmission of disease is taken as the direct contact between the individuals. However, many diseases are transmitted to the susceptible individuals in the species not only through direct contact, but also indirectly from environment. Das et al [4] proposed on a prey-predator model with disease in prey that spread by contact and external sources, included Holling type II as a functional response. Dobson [5] studied the situations where the behavior of the infected host is modified by the action of a parasite. the infected prey may become weaker and less active so that they may be easily caught by the predator (Moore , [6])

In this chapter a prey-predator model involving *SIS* infectious disease in prey species is proposed and analyzed. It is assumed that the disease transmitted within the prey population by contact and an external sources .The existence, uniqueness and boundedness of the solution are discussed. The existence and the stability analysis of all possible equilibrium points are studied. Finally, the global dynamics of the model is carried out analytically as well as numerically.

The mathematical model:

In this section, a prey-predator system involving an *SIS* epidemic disease in prey population is proposed for study. In the presence of disease, the prey population is divided into two classes: the susceptible individuals $S(t)$ and the infected individuals $I(t)$, here $S(t)$ represents the density of susceptible individuals at time t while $I(t)$ represents the infected individuals at time t. The prey population grows logistically with intrinsic growth rate r and environmental carrying capacity $K(K>0)$. The existence of disease may causes death in the infected prey with positive death rate d_1. The predator species consumes the prey species (susceptible as well as infected) according to modified Holling type-II functional response with predation rate $P_1>0$ and $P_2>0$ respectively and half – saturation constant m, however it converts the food from susceptible and infected prey with a conversion rates $e_1>0$ and $e_2>0$ respectively. Finally, in the absence of prey species the predator species decay exponentially with a natural death rate $d_2>0$. Now in order to formulate our model, the following assumptions are adopted:

Consider a prey-predator system in which the density of prey at time t is denoted by $N(t)$ while the density of predator species at time t is denoted by $Y(t)$. We impose the following assumptions:

1. Only the susceptible prey can reproduce logistically, however the infected prey can't reproduce but still has a capability to compete with the other prey individuals for carrying capacity.
2. The susceptible prey becomes infected prey due to contact between both the species as well as an external sources for the infection with the contact infection rate constant $\beta \succ 0$ and external infection rate $C \succ 0$. However, the infected prey recover and return to the susceptible prey with a recover rate constant $\gamma \succ 0$. According to the above hypothesis the dynamics of a prey-predator model involving an *SIS* epidemic disease in prey population can be describe by the following set of nonlinear differential equations:

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - (\beta I + C)S + \gamma I - \frac{P_1 SY}{m + S + nI}$$
$$\frac{dI}{dt} = (\beta I + C)S - d_1 I - \gamma I - \frac{P_2 IY}{m + S + nI} \qquad \ldots(1)$$
$$\frac{dY}{dt} = -d_2 Y + \frac{e_1 P_1 SY + e_2 P_2 IY}{m + S + nI}$$

here
$n > 0$ represents the preference rate constant and $S(0) \geq 0$, $I(0) \geq 0$ and $Y(0) \geq 0$. Obviously the interaction functions of the system (1) are continuous and have continuous partial derivatives on the region. $R_+^3 = \{(S, I, Y) \in R^3 : S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0\}$. Therefore these functions are Lipschitzian on R_+^3, and hence the solution of the system (1) exists and is unique. Further, in the following theorem, the boundedness of the solutions of the system (1) in R_+^3 is established**.**

Theorem (1): All the solutions of the system (1) are uniformly bounded.
proof: Let $W = S + I + Y$ then

$$\frac{dW}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dY}{dt}$$
$$\frac{dW}{dt} \leq S\left(r\left(1 - \frac{S}{K} - \frac{I}{K}\right)\right) - d_1 I - d_2 Y$$
$$\frac{dW}{dt} \leq S\left(r\left(1 - \frac{S}{K}\right)\right) - d_1 I - d_2 Y$$
$$\frac{dW}{dt} \leq S(r+1) - S - d_1 I - d_2 Y$$

Let $\hat{K} = \max\{S(0), K\}$ and $d = \min\{1, d_1, d_2\}$ then

$$\frac{dW}{dt} \leq \hat{K}(r+1) - dW$$

thus $t \to \infty$ it is easy to verify that $0 \leq W \leq \frac{\hat{K}(r+1)}{d}$

Hence all solutions of system (1) are uniformly bounded and therefore we have finished the proof

The stability analysis of system (1):

In this section the existence and stability analysis of all feasible equilibrium points of system (1) are studied.

1.The trivial equilibrium point , which denoted by $E_1 = (0,0,0)$, always exists.

2. The predator free equilibrium point that denoted by $E_2 = (\hat{S}, \hat{I}, 0)$, where

$$\hat{S} = \frac{-B_1 + \sqrt{B_1{}^2 - 4B_2}}{2} \text{ and } \hat{I} = \frac{C\hat{S}}{d_1 + \gamma - \beta\hat{S}} \qquad \ldots(2a)$$

here

$B_1 = -\frac{Cr + r(\gamma + d_1 + K\beta)}{r\beta} \prec 0$ and $B_2 = \frac{rK(d_1+\gamma) - Cd_1K}{r\beta}$, exists uniquely in the $Int.R_+^2$ of $SI-$plane provided that the following condition holds

$$\beta\hat{S} < d_1 + \gamma < \frac{Cd_1}{r} \qquad \ldots(2b)$$

3. However the positive equilibrium point that denoted by $E_3 = (S^*, I^*, Y^*)$ where

$$S^* = \frac{-d_2(m + nI^*) + e_2P_2I^*}{d_2 - e_1P_1}, d_2 \neq e_1P_1, Y^* = \left[(\beta I^* + C)S^* - d_1I^* - \gamma I^*\right]\frac{m + S^* + nI^*}{P_2I^*} \ldots (3a)$$

While

I^* represents a positive root of the following third order polynomial equation $A_3I^3 + A_2I^2 + A_1I + A_0 = 0$ here $A_0 = \frac{d_2^2 m^2 CP_1}{H}$ where $H = d_2 - e_1P_1$

$$A_1 = \frac{d_2 m}{H}\left[d_2\left(2nCP_1 - m\left(P_1\beta + \frac{rP_2}{K}\right)\right) + P_1H(\gamma + d_1) + p_2H(r - c)\right]$$

$$A_2 = \frac{d_2mP_2r}{KH}\left(d_2(2n - H)\right) - P_2\beta H + 2P_1\beta(nd_2 + e_2P_2) + P_2\gamma(e_1P_1 - d_2) - \frac{e_2P_2P_1}{H}(Ce_2P_2 + \gamma H + d_1H) + \frac{P_1d_2}{H}\left(d_2n^2C + P_1\gamma Hn + nd_1H\right) + P_2(C - r)(e_2P_2 - d_2n)$$

$$A_3 = P_2\beta(e_2P_2 - d_2n) + \frac{P_1\beta e_2P_2}{H}(e_2P_2 - 2d_2n) + \frac{P_2r}{KH}\left(d_2^2n^2 - e_2^2P_2^2 + (e_2P_2 - d_2n)H\right) + \left(\frac{P_1\beta d_2^2n^2}{H}\right)$$

exists uniquely in the $Int.R_+^3$ if and only if the following conditions are hold.

$$\left.\begin{array}{l} A_3 \succ 0, A_2 \succ 0 \quad and \quad A_0 \prec 0 \\ or \\ A_3 \prec 0, A_2 \prec 0 \quad and \quad A_0 \succ 0 \\ or \\ A_3 \succ 0, A_1 \prec 0 \quad and \quad A_0 \prec 0 \\ or \\ A_3 \prec 0, A_1 \succ 0 \quad and \quad A_0 \succ 0 \end{array}\right\} \quad \text{…(3b)}$$

$$e_2 P_2 I^* > d_2 \left(m + nI^*\right) \text{with } d_2 > e_1 P_1 \quad \text{…(3c)}$$

Or $d_2\left(m+nI^*\right) > e_2 P_2 I^*$ with $e_1 P_1 > d_2$

Now the stability analysis of the above feasible equilibrium points of system (1) are studied analytical with help of Linearization method. Note that it is easy to verify that, the Jacobian matrix of system (1) at the trivial equilibrium point $E_1 = (0,0,0)$ can be written in the form:

$$J(E_1) = \begin{pmatrix} r-C & \gamma & 0 \\ C & -d_1-\gamma & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

Thus the characteristic equation of $J(E_1)$ can be written as

$$\left[\lambda_1{}^2 + (-r + C + d_1 + \gamma)\lambda_1 + (Cd_1 - (d_1+\gamma)r)\right](-d_2 - \lambda)_1 = 0$$

Accordingly, the eigenvalues of $J(E_1)$satisfy the following relations:

$$\lambda_{1S} + \lambda_{1I} = r - C - d_1 - \gamma, \lambda_{1S}.\lambda_{1I} = Cd_1 - (d_1+\gamma)r, \lambda_{1Y} = -d_2 \prec 0 \quad \text{… (4a)}$$

Thus E_1 is locally asymptotically stable provided that the following condition holds $r - C < d_1 + \gamma < \frac{Cd_1}{r}$ …(4b)

However, it is saddle point otherwise.

The Jacobian matrix of system (1) at the predator free equilibrium point E_2 can be written as:

$$\text{J}(E_2) = \begin{pmatrix} r\left(1 - \frac{2\hat{S}+\hat{I}}{K}\right) - \left(\beta\hat{I} + C\right) & \frac{-r\hat{S}}{K} - \beta\hat{S} + \gamma & \frac{-P_1\hat{S}}{m+\hat{S}+n\hat{I}} \\ \beta\hat{I} + C & \beta\hat{S} - d_1 - \gamma & \frac{-P_2\hat{I}}{m+\hat{S}+n\hat{I}} \\ 0 & 0 & -d_2 + \frac{e_1P_1\hat{S} + e_2 p_2 \hat{I}}{m+\hat{S}+n\hat{I}} \end{pmatrix}$$

Consequently, the characteristic equation can be written as

$$\left(\lambda_2{}^2 - T\lambda_2 + D\right)\left[-d_2 + \frac{e_1P_1\hat{S} + e_2P_2\hat{I}}{m+\hat{S}+n\hat{I}} - \lambda_2\right] = 0 \text{ .here}$$

$$T = r\left(1-\frac{2\hat{S}+\hat{I}}{K}\right) - \left(\beta\hat{I}+C\right) + \left(\beta\hat{S}-d_1-\gamma\right)$$

$$D = \left[r\left(1-\frac{2\hat{S}+\hat{I}}{K}\right) - \left(\beta\hat{I}+C\right)\right]\left(\beta\hat{S}-d_1-\gamma\right) - \left(\beta\hat{I}+C\right)\left(\frac{-r\hat{S}}{K}-\beta\hat{S}+\gamma\right)$$

Clearly the eigenvalues of this Jacobian matrix satisfy the following relationships:

$T = \lambda_{2S}+\lambda_{2I}$; $D = \lambda_{2S}.\lambda_{2I}$,and $\lambda_{2Y} = -d_2 + \dfrac{e_1P_1\hat{S}+e_2P_2\hat{I}}{m+\hat{S}+n\hat{I}}$ …(5a)

Accordingly the equilibrium point E_2 is locally asymptotically stable provided that:

$r\left(1-\dfrac{2\hat{S}+\hat{I}}{K}\right) < \left(\beta\hat{I}+C\right), \dfrac{K\gamma}{r+\beta K} < \hat{S}, \dfrac{e_1P_1\hat{S}+e_2P_2\hat{I}}{m+\hat{S}+n\hat{I}} < d_2$ …(5b)

However, it is a saddle point otherwise.

The Jacobian matrix of system (1) at E_3 is given by $J(E_3) = (a_{ij})_{3\times 3}$,

where:

$a_{11} = r\left(1-\dfrac{2S^*+I^*}{K}\right) - M_1 - \dfrac{P_1Y^*\left(m+nI^*\right)}{M_2^2}$, $a_{12} = \dfrac{-rS^*}{K} - \beta S^* + \gamma + \dfrac{nP_1S^*Y^*}{M_2^2}$,

$a_{13} = \dfrac{-P_1S^*}{M_2} < 0$

$a_{21} = M_1 + \dfrac{P_2I^*Y^*}{M_2^2} > 0$, $a_{22} = \beta S^* - d_1 - \gamma - \dfrac{P_2Y^*(m+S^*)}{M_2^2}$ $a_{23} = \dfrac{-P_2I^*}{M_2} < 0$

$a_{31} = \dfrac{me_1P_1-(e_2P_2-ne_1P_1)I^*}{M_2^2}Y^*$ $a_{32} = \dfrac{me_2P_2+(e_2P_2-ne_1P_1)S^*}{M_2^2}Y^*$; $a_{33} = 0$

Where

$M_1 = \beta I^* + C$ and $M_2 = m + S^* + nI^*$

Then the characteristic equation of $J(E_3)$ can be written as:

$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$ …(6a)

$B_1 = -(a_{11}+a_{22}), B_2 = a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}$

$B_3 = a_{31}(a_{13}a_{22} - a_{12}a_{23}) + a_{32}(a_{11}a_{23} - a_{13}a_{21})$

According to Routh-Hurwitz criterion the equilibrium point E_3 is locally asymptotically stable .Provided that $B_1 \succ 0$, $B_3 \succ 0$ and $\Delta = B_1B_2 - B_3 \succ 0$,Hence straight forward computation show That equilibrium point E_3 is locally asymptotically stable provided that

1. $1 \prec \dfrac{2S^*+I^*}{K}$ **…** (6b)

and $\beta S^* \prec d_1 + \gamma$ …(6c)

2. $I^* \prec \dfrac{me_1P_1}{e_2P_2 - ne_1P_1}$ Or $S^* \prec \dfrac{me_2p_2}{ne_1p_1 - e_2p_2}$ …(6d)

3. $\left(d_1+\gamma-\beta S^*\right)P_1 S^* \succ \left(\left(\frac{r}{K}+\beta\right)-\gamma\right)P_2 I^*$ ...(6e)

4. $\left(d_1+\gamma+\frac{P_2 Y^*\left(m+S^*\right)}{M_2^2}-\beta S^*\right)P_2 I^* \succ \left(M_1+\frac{P_2 I^* Y^*}{M_2^2}\right)P_1 S^*$ …(6f)

Clearly, the condition (6b)-(6d) guarantee that a_{11}, a_{12} and a_{22} are negative while a_{31} and a_{32} are positive and hence $B_1 \succ 0$ .Further the conditions (6b)-(6e) guarantee that $B_3 \succ 0$. Finally the condition (6b)-(6f) guarantee that $B_1 B_2 - B_3 \succ 0$

Theorem (2):

Assume that E_1 is a locally asymptotically stable point in R_+^3 then E_1 is globally asymptotically stable in the sub region of R_+^3 that given by:

$\Gamma_1 = \left\{(S,I,Y)\in R_+^3 : K \prec S+I, Y \geq 0\right\}$

Proof: Consider the following positive definite function:

$L_1(S,I,Y) = S+I+Y$ … (7)

Clearly, $L : R_+^3 \to R$ is continuously differentiable function so that $L_1(0,0,0)=0$ and $L_1(S,I,Y)>0$ for all $(S,I,Y)\in R_+^3$ with $(S,I,Y)\neq(0,0,0)$. Therefore by differentiating this function with respect to the variable t we get that: $\frac{dL_1}{dt}=\frac{dS}{dt}+\frac{dI}{dt}+\frac{dY}{dt}$ Substituting the value of $\frac{dS}{dt},\frac{dI}{dt}$ and $\frac{dY}{dt}$ in this equation and then simplifying the resulting terms we obtain: $\frac{dL_1}{dt}=1-\frac{S+I}{K}$ Clearly , $\frac{dL_1}{dt} \prec 0$ on Γ_1 and hence the function L_1 is a Lyapunov function. Thus E_1 is globally asymptotically stable on the sub region Γ_1, and hence the proof is finished. ■

Theorem (3): Assume that E_2 is a locally asymptotically stable point in R_+^3 then E_2 is globally asymptotically stable on the sub region of R_+^3 that satisfy the following conditions:

$G_2^{\,2} < 4G_1 G_3$ …(8a)

$\left(\sqrt{G_1}U_1-\sqrt{G_3}U_2\right)^2 \succ \frac{Y}{m+S+nI}\left(SP_1N_1+IP_2N_2\right)$ …(8b)

here $G_1 = C-r+\left(\frac{r}{K}+\beta\right)\hat{I}+\frac{r}{K}\hat{S}$, $G_2=\gamma+C-\left(\frac{r}{K}+\beta\right)S+\beta\hat{I}$,

$G_3 = d_1+\gamma-\beta S$,

$U_1 = S-\hat{S}$ and $U_2 = I-\hat{I}$, $N_1=\hat{S}+e_1$ and $N_2=\hat{I}+e_2$

Proof: Consider the following positive definite function:
$L_2 = \frac{(S-\hat{S})^2}{2} + \frac{(I-\hat{I})^2}{2} + Y$
Clearly, $L_2 : R_+^3 \rightarrow R$ is continuously differentiable function so that $L_2(\hat{S},\hat{I},0) = 0$ and $L_2(S,I,0) \succ 0$ for all $(S,I,Y) \in R_+^3$ with $(S,I,Y) \neq (0,0,0)$.

Therefore by differentiating this function with respect to the variable t we get: $\frac{dL_2}{dt} = (S-\hat{S})\frac{dS}{dt} + (I-\hat{I})\frac{dI}{dt} + \frac{dY}{dt}$ .Substituting the value of $\frac{dS}{dt}, \frac{dI}{dt}$ and $\frac{dY}{dt}$ in this equation and then simplifying the resulting terms we obtain:

$$\frac{dL_2}{dt} \leq -G_1 U_1{}^2 + G_2 U_1 U_2 - G_3 U_2{}^2 + \frac{Y}{m+S+nI}(SP_1N_1 + IP_2N_2)$$

So, by using condition (8a) we obtain that:

$$\frac{dL_2}{dt} \leq -\left(\sqrt{G_1}U_1 - \sqrt{G_3}U_2\right)^2 + \frac{Y}{m+S+nI}(SP_1N_1 + IP_2N_2)$$

Now according to condition (8b) it is easy to verify that $\frac{dL_2}{dt} < 0$, and hence L_2 is a Lyapnuov function. Thus E_2 is globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions. ■

Theorem (4): Assume that E_3 is a locally asymptotically stable point in R_+^3 then E_3 is globally asymptotically stable on the sub region of R_+^3 that satisfy the following conditions:

$$G_5{}^2 < 4G_4G_6 \quad \text{...(9a)}$$

$$\left(\sqrt{G_4}U_3 - \sqrt{G_6}U_4\right)^2 \succ \frac{Y}{m+S+nI}(SP_1N_3 + IP_2N_4) + \frac{Y^*}{m+S^*+nI^*}\left(P_1S^*N_5 + P_2I^*N_6\right) \text{...(9b)}$$

here

$G_4 = C - r + \left(\frac{r}{K} + \beta\right)I^* + \frac{r}{K}S^*$, $G_5 = -\left(\frac{r}{K} + \beta\right)S + \beta I^* + C + \gamma$, $G_6 = d_1 + \gamma - \beta S$, $U_3 = S - S^*$ and $U_4 = I - I^*$, $N_3 = S^* + e_1Y^*$, $N_4 = I^* + e_2Y^*$, $N_5 = S + e_1Y^*$ and $N_6 = I + e_2Y^*$

Proof: Consider the following positive definite function:
$L_3 = \frac{(S-S^*)^2}{2} + \frac{(I-I^*)^2}{2} + \frac{(Y-Y^*)^2}{2}$ .Clearly, $L_3 : R_+^3 \rightarrow R$ is continuously differentiable function so that $L_3(S^*,I^*,Y^*) = 0$ and $L_3(S,I,Y) \succ 0$ for all $(S,I,Y) \in R_+^3$ with $(S,I,Y) \neq (0,0,0)$. Therefore by differentiating this function with respect to the variable *t* we get:

$\frac{dL_3}{dt} = \left(S - S^*\right)\frac{dS}{dt} + \left(I - I^*\right)\frac{dI}{dt} + \left(Y - Y^*\right)\frac{dY}{dt}$ .Substituting the value of $\frac{dS}{dt}, \frac{dI}{dt}$ and $\frac{dY}{dt}$ in this equation and then simplifying the resulting terms we obtain :

$$\frac{dL_3}{dt} \le -G_4U_3^2 + G_5U_3U_4 - G_6U_4^2 + \frac{Y}{m+S+nI}\;\frac{Y^*}{m+S^*+nI^*}\left(P_1S^*N_5 + P_2I^*N_6\right)$$

So, by using condition (2.9a) we obtain that:

$$\frac{dL_3}{dt} \le -\left(\sqrt{G_4}U_3 - \sqrt{G_6}U_4\right)^2 + \frac{Y}{m+S+nI}\left(SP_1N_3 + IP_2N_4\right) + \frac{Y^*}{m+S^*+nI^*}\left(P_1S^*N_5 + P_2I^*N_6\right)$$

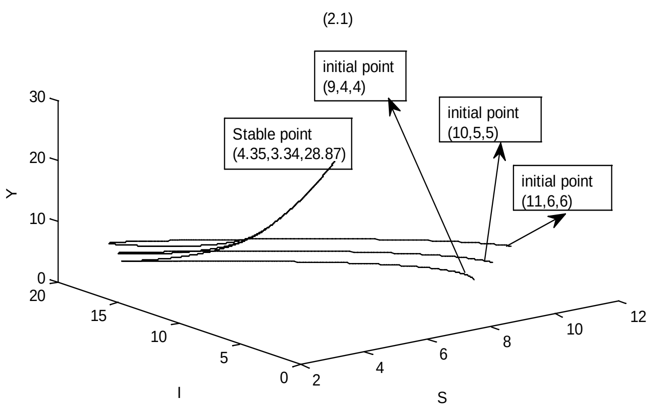
Now according to condition (9b) it is easy to verify that $\frac{dL_3}{dt} < 0$, and hence L_2 is a Lyapnuov function. Thus E_3 is globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions. ■

Numerical Simulation*:*

In this section the dynamical behavior of system (1) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (1) and second confirm our obtained analytical results. Now for the following set of hypothetical parameters values:

$$r = 1, K = 200, \beta = 0.2, C = 0.1, \gamma = 0.4, P_1 = 1, P_2 = 1,$$
$$d_1 = 0.1, n = 1, m = 50, d_2 = 0.1, e_1 = 0.75, e_2 = 0.75 \quad \ldots(10)$$

The trajectory of the system (1) is drawn in the Fig.(1)for different initial points.



Figure(2.1): Phase plot of system (1) starting from different initial points.

In the above figure, system (1) approaches asymptotically to the stable coexistence equilibrium point starting from different initial points. Note that in time series figures, we will use throughout this section that: blue color for describing the trajectory of S ;

green color for describing the trajectory of I ; red color for describing the trajectory of Y .
Now in order to discuss the effect of varying the intrinsic rate r on the dynamical behavior of system (1), the system (1) is solved for different values of the mortality rates $r = 0.02, 0.1, 0.5$ keeping other parameters fixed as given in Eq (10).

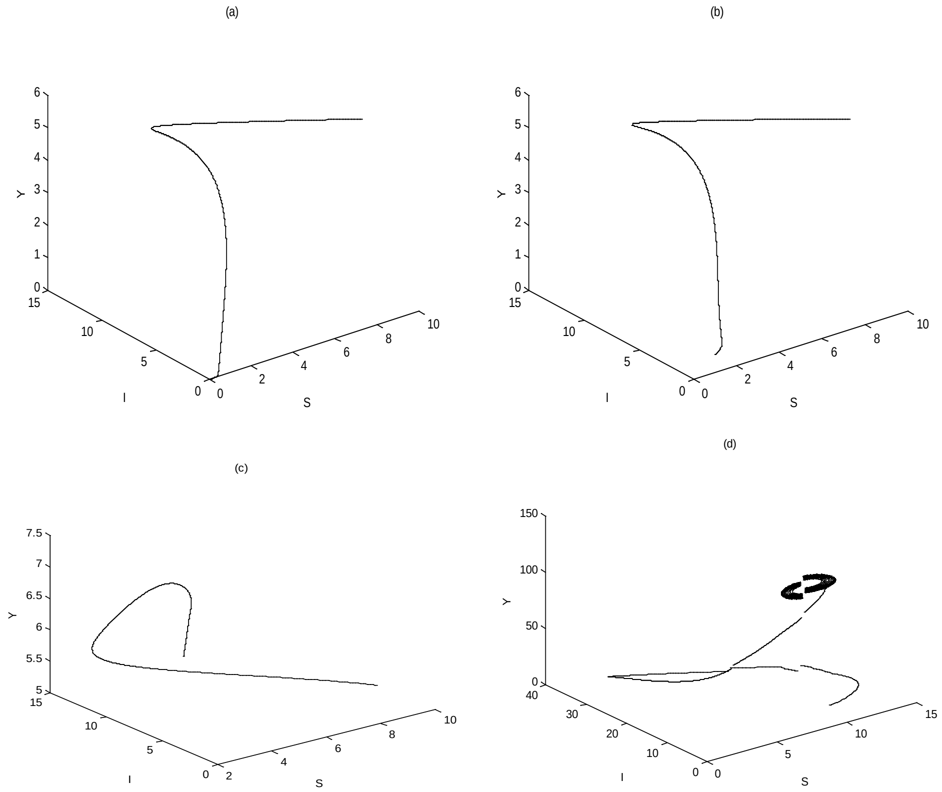


Figure (2): Phase plots of system (1). for the data given by Eq.(10). (*a*)system (1) approaches asymptotically to E_0 then. $r = 0.02$ (*b*) system (1) approaches asymptotically to predator free equilibrium point on SI-plane when *r=0.1* (*c*) system (1) approaches asymptotically to coexistence equilibrium point when *r*=0.5 (d) system (1) approaches to periodic attractor in the interior of R_+^3 when r = 2.5.

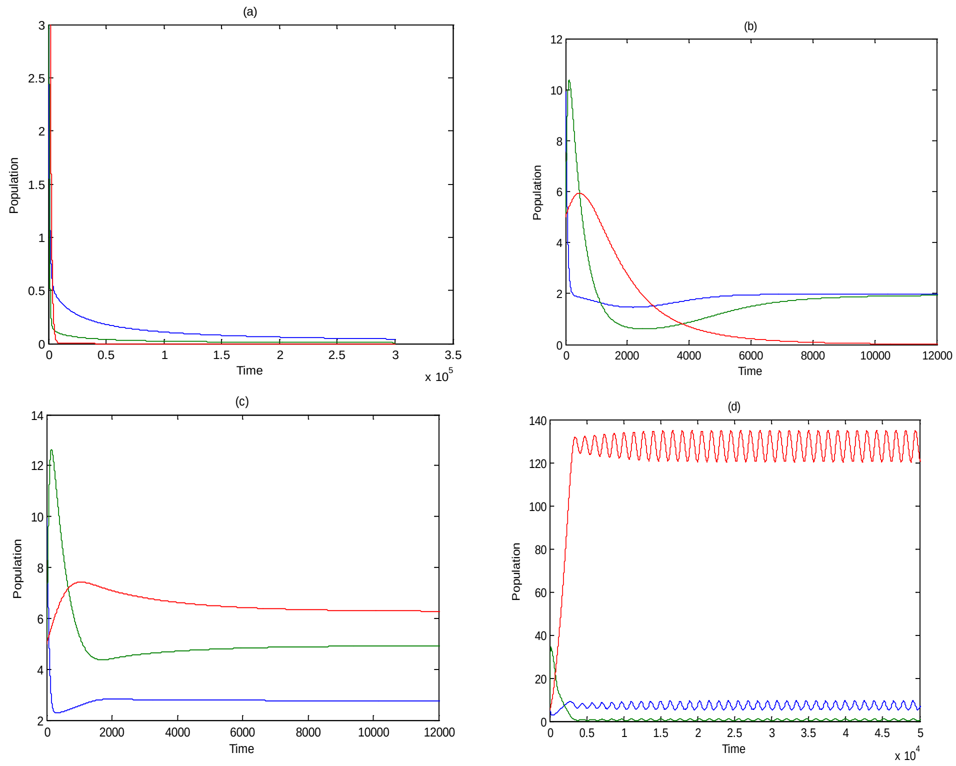


Figure (3): Time series for the solution of system (1). (*a*) time series for the attractor in Fig.(2*a*) ,(*b*) time series for the attractor in Fig.(2b), (c) time series for the attractor in Fig. (2*c*) ,(*d*) time series for the attractor in Fig. (2*d*)

The effect of contact infection rate β on the dynamic of system (2.1)studied and the trajectories of system (1) are drawn in Fig. (4a)-(4c) for the values β = 0.05, 0.5, respectively, keeping other parameters fixed as given in Eq.(10).

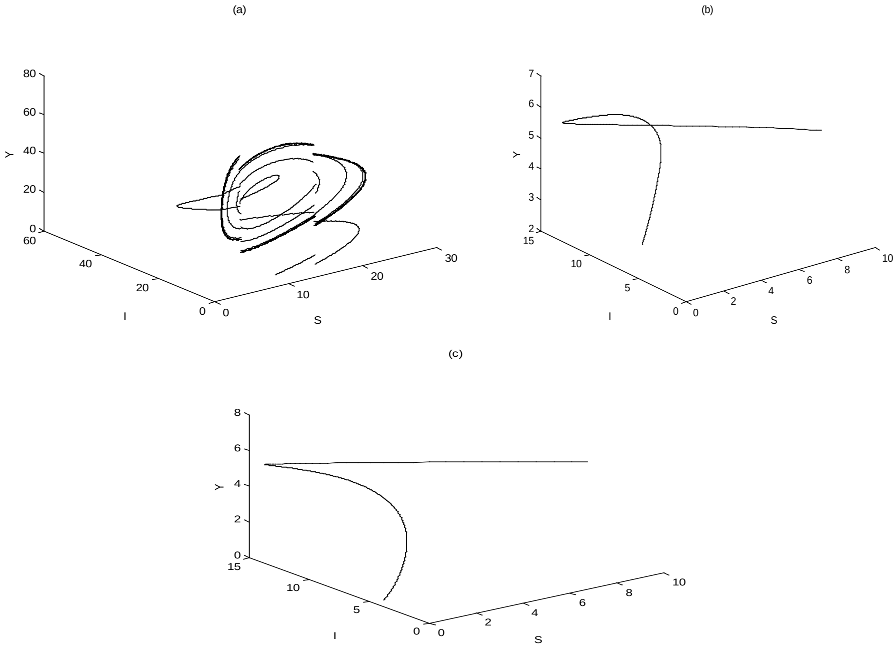


Figure (4): Phase plots of system (1). for data given in Eq.(10)(a) system (1) approaches to periodic attractor for β =0.05, (b) system (2.1) approaches asymptotically to coexistence equilibrium point for β =0.5, (c) system (1) approaches asymptotically to predator free equilibrium point on SI-plane for β =1.

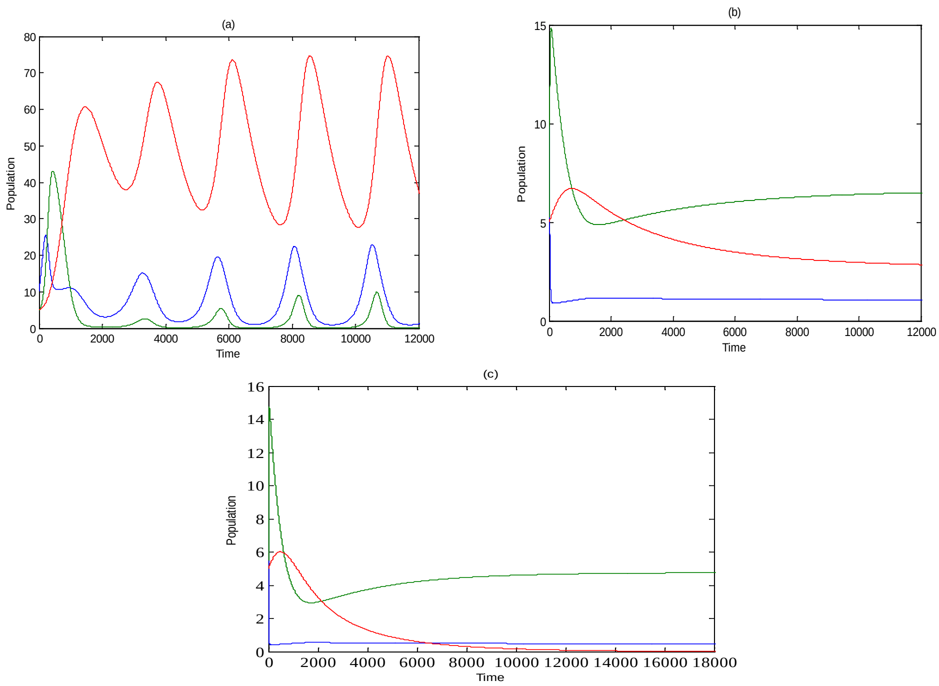


Figure (5): Time series for the solution of system (1) for the data given in Eq.(10) (a) time series for the attractor in Fig. (4a) (b) time series for the attractor in Fig. (4b) , (c) time series for the attractor in Fig. (4*c*)

The effect of varying recover rate γ on the dynamic behavior of system (1) is studied and the trajectories of system (1) are drawn in Fig. (6a)-(6c) for the values γ = 0.01,0.1,0.8 respectively keeping other parameters fixed as given in Eq. (10).

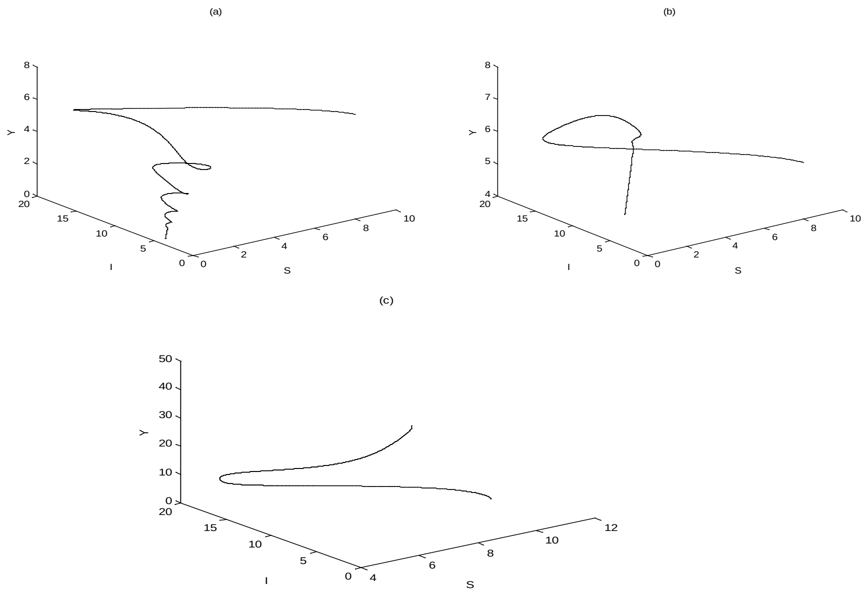


Figure (6): Phase plots of system (1) for the data given in Eq.(10)(*a)* system (1) approaches asymptotically to predator free equilibrium point on SI- plane for $\gamma = 0.01$ (*b*) system (1) approaches asymptotically to coexistence equilibrium point for γ =0.1 (c) system (1) approaches asymptotically to coexistence equilibrium point for γ =0.8

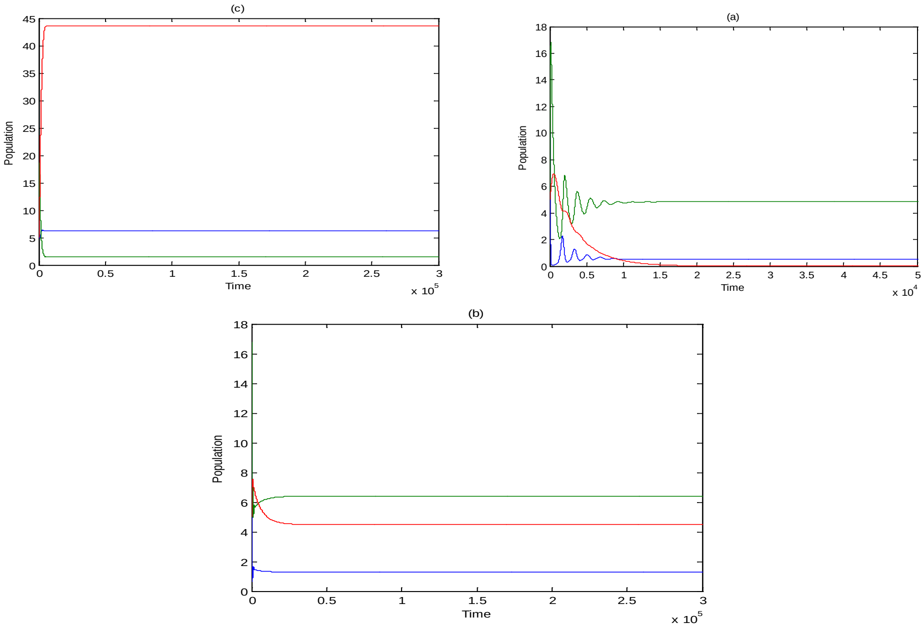


Figure (7): Time series for the solution of system (1). (*a*) time series for the attractor in Fig. (6*a*) , (*b*) time series for the attractor in Fig. (6*b*) (*c*) time series for the attractor in Fig. (6*c*)

The effect of varying attack rate P_2 of infected prey species on the dynamics of system (1) is studied and the trajectories of system (1) are drawn in Fig. (8a)-(8c) for the values $P_2 = 0.01, 0.3, 0.8$ respectively

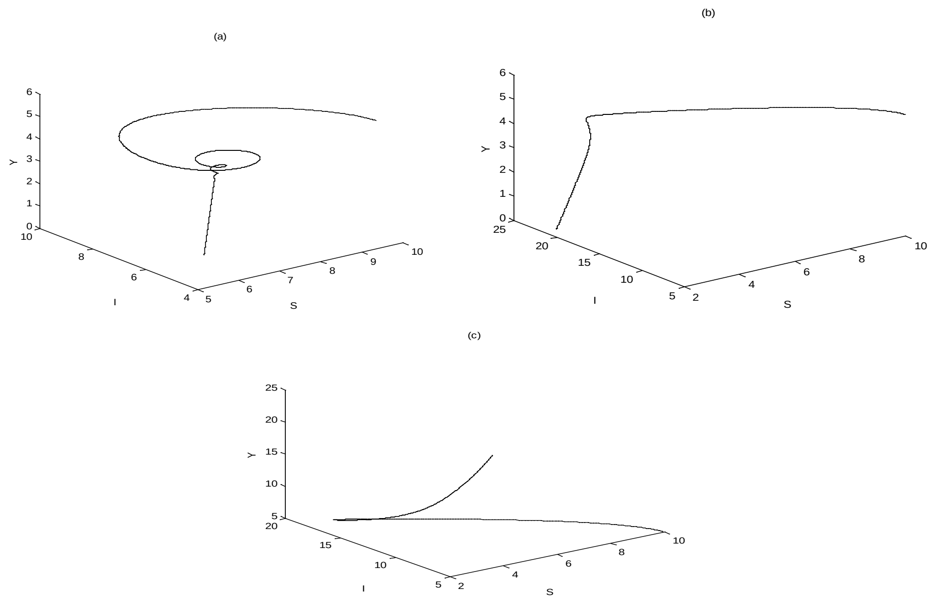


Figure (8): Phase plots of system (1)For the data given in Eq.(10)(a) system (1) approaches asymptotically to predator free equilibrium point on SI-plane for P_2 =0.01 (b) system (1) approaches asymptotically to predator free equilibrium point on SI-axis

for P_2 =0.3.(c) system (1) approaches asymptotically to coexistence equilibrium point for P_2 =0.8.

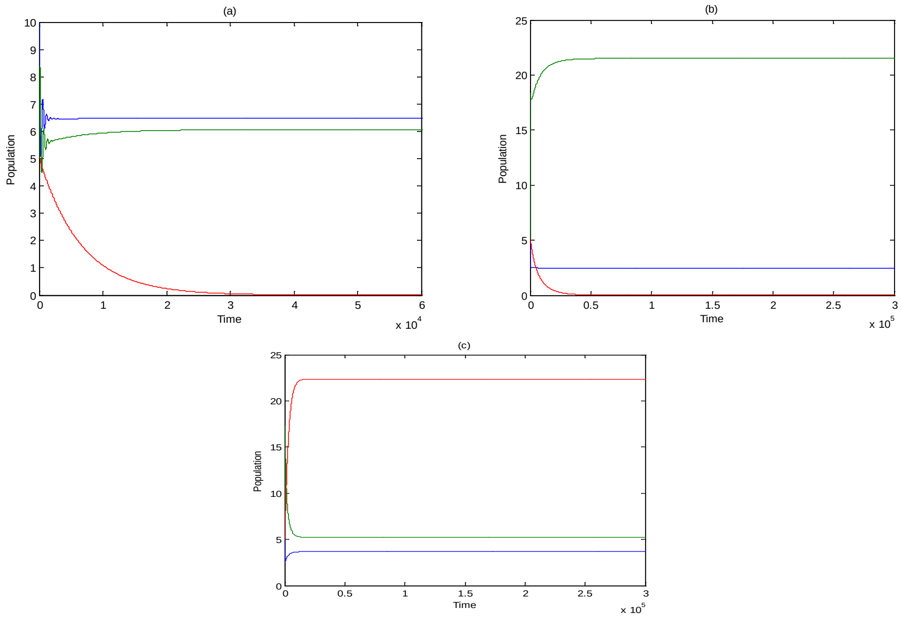


Figure (9): Time series for the solution of system (1). (a) time series for the attractor in Fig. (8a) , (b) time series for the attractor in Fig.(8b), (c) time series for the attractor in Fig. (8c)

The effect of varying half-saturation constant m the dynamic behavior of system (1) is studied and the trajectories of system (1) are drawn in Fig. (10a)-(10c) for the values m = 10,60 respectively.

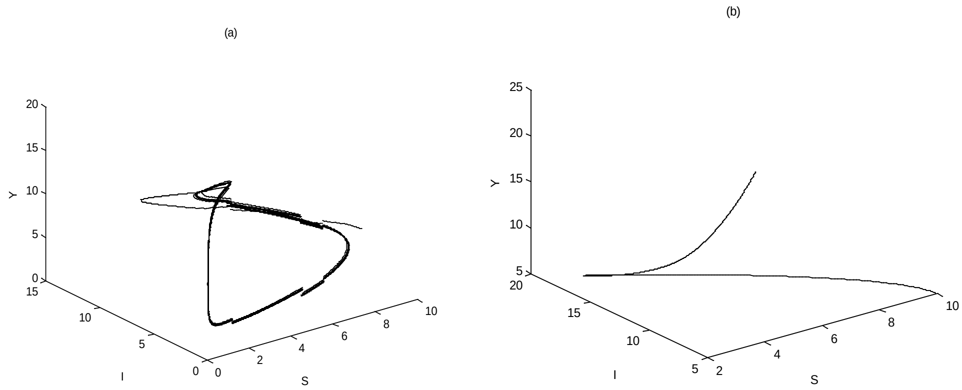


Figure (10): Phase plots of system (1)for the data given in Eq.(10) (a) system (1) approaches to periodic attractor for m=10, (b) system (1) approaches asymptotically to coexistence equilibrium point for m=60

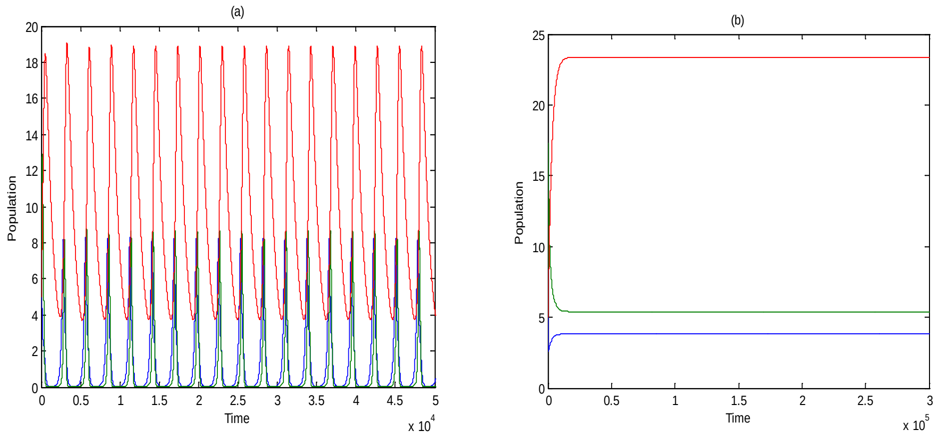


Figure (11*):* Time series for the solution of system (1). (*a*) time series for the attractor in Fig. (10*a*) (b) time series for the attractor in Fig. (10b) , (*c*) time series for the attractor in Fig. (10c)

The effect of varying death rate of predator d_2. dynamics of system (1) is studied and the trajectories of system (1) are drawn in Fig. (12a)-(12c) for the values d_2 = 0.01,0.2,0.7 respectively. while their time series are drawn in Fig.(13a)-(13c) respectively.

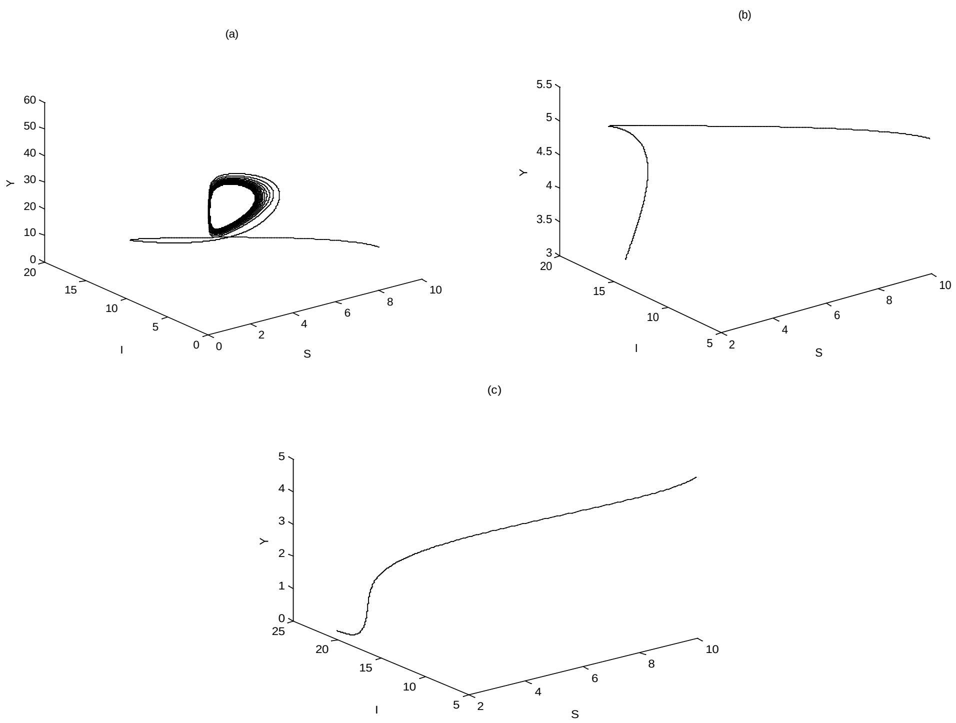
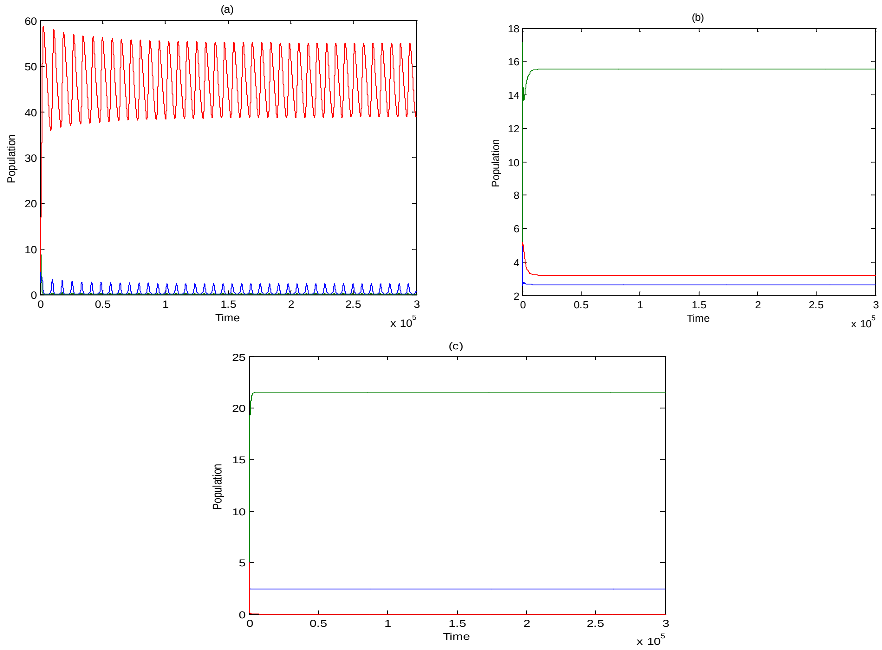
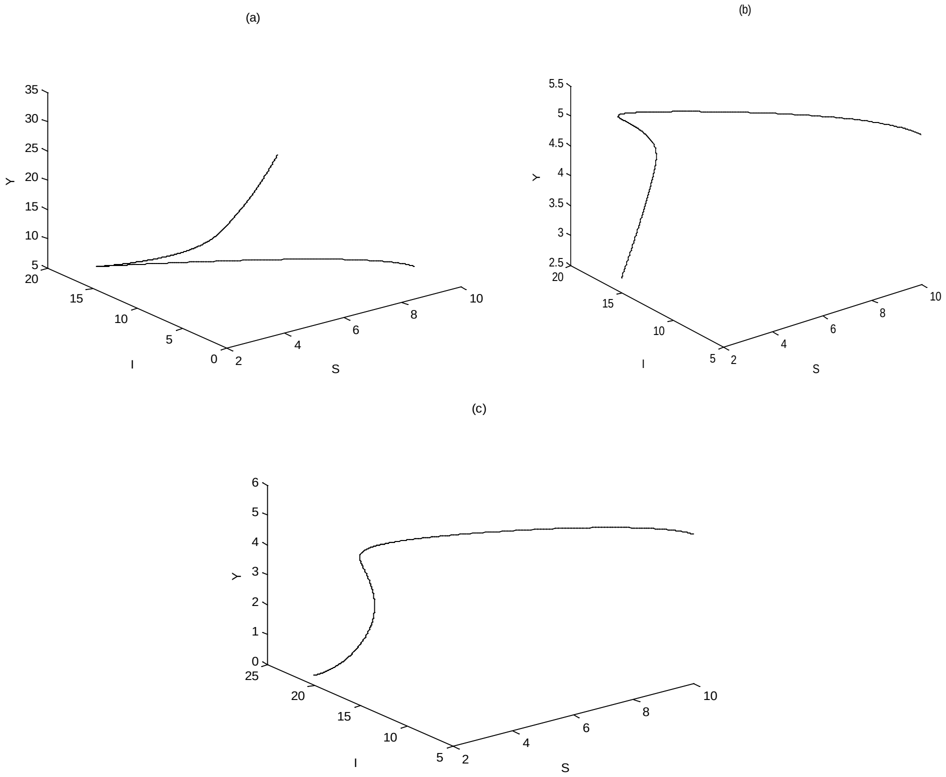


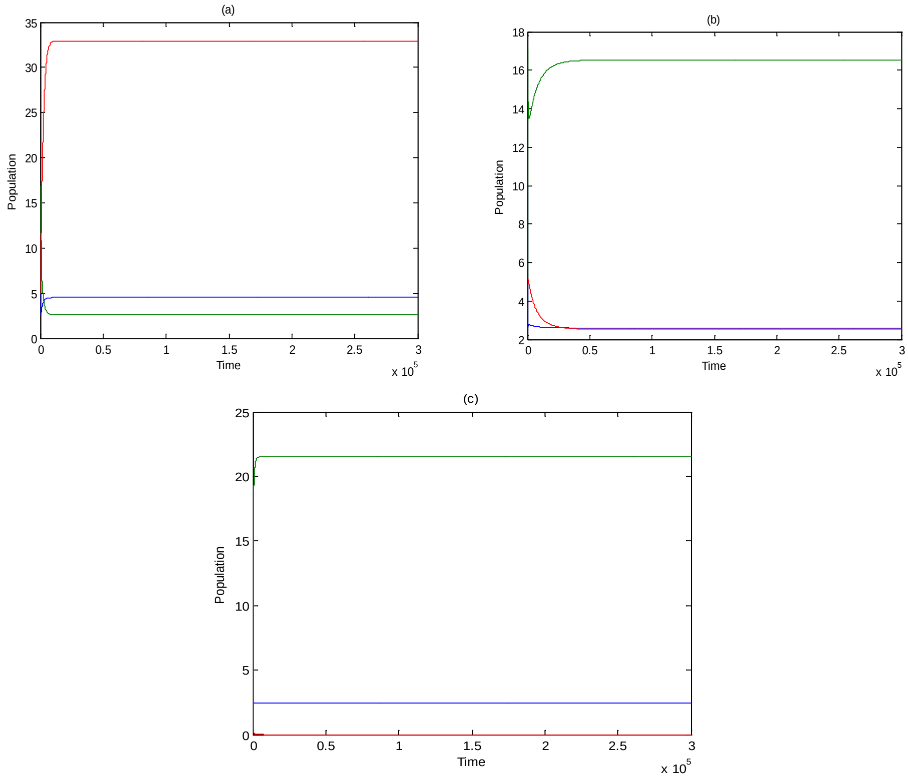
Figure (12): Phase plots of system (1)for the data given in Eq. (10)(*a*) system (1) approaches to periodic attractor for d_2 =0.01, (b) system (1) approaches asymptotically to coexistence equilibrium point for d_2 =0.2 (*c*) approaches asymptotically to predator free equilibrium point on SI-plane for d_2 =0.7



The effect of varying conversion rates e_2 of infected prey species on the dynamic behavior of system (1) is studied and the trajectories of system (1) are drawn in Fig. (14a)-(14c) for the values e_2 = 0.9,0.3,0.01 keeping other parameters fixed as given in Eq. (10).



Figure(14): Phase plots of system (1) for the data given by Eq. (10) (a) system (1) approaches to asymptotically to coexistence equilibrium point for e_2 =0. 9, ((b) system (1) approaches asymptotically to coexistence equilibrium point for e_2 =*0.3*, (*c*) approaches asymptotically to predator free equilibrium point on SI -Plane for e_2 =0.01.



Figure(15): Time series for the solution of system (1). (*a*) time series for the attractor in Fig. (14a) (b) time series for the attractor in Fig. (14b) (*c*) time series for the attractor in Fig. (14c).

Discussion and conclusion:

In this chapter, we proposed and analyzed an eco-epidemiological model that describe the dynamical behavior of a prey-predator model with linear functional response. The model consisting of three non-linear differential equations that describe the dynamics of three different populations namely predator Y, susceptible prey S, infected prey I. The boundedness of the system (1) has been discussed. The conditions for existence and stability of each equilibrium points are obtained. To understand the effect of varying each parameter on the dynamical behavior of the system a numerically simulation has been used and the obtained results can be summarized as follow

1. Decreasing the intrinsic growth rate *r* in the range $r \leq 0.02$ causes that extinction in all populations and the system (1) approaches asymptotically to the vanishing equilibrium point E_1. However for $0.02 \prec r \leq 0.24$ it is observed that the system (1) approaches asymptotically to the predator free equilibrium point E_2. More over increasing the intrinsic growth rate in the range $0.24 \prec r \prec 2.4$ causes to coexistence of all populations and the system (1) approaches asymptotically stable to E_3. Finally, for $r \geq 2.4$ the coexistence equilibrium point E_3 loses its stability and the system approaches asymptotically to the periodic dynamic in the $Int.R_+^3$

2. Decreasing the values of contact infection rate β in the range $\beta \leq 0.08$ leads to periodic dynamic in the $Int.R_+^3$. However for $0.08 \prec \beta \prec 0.7$ it is observed that the system (1) approaches asymptotically stable to the coexistence point E_3. Finally increasing β in the range $0.7 \leq \beta \leq 1$ causes extinction in predator species and the system (1) approaches asymptotically to the predator free equilibrium point E_2 .
3.Decreasing the recover rate γ in the range $\gamma \leq 0.055$ causes extinction in predator species and the system (1) approaches asymptotically to the predator free equilibrium point E_2. However, as γ increases the trajectory of system (1) approaches asymptotically to the coexistence equilibrium point
4. Decreasing values of half-saturation constant *m* in the range $m \leq 23$, leads to the periodic dynamic in the $Int.R_+^3$. However, as *m* increases the trajectory of system (1) approaches asymptotically to the coexistence equilibrium point
5. For the values of the death rate of predator d_2 in the range $d_2 \leq 0.02$ the system (1) periodic dynamic in the $Int.R_+^3$. However for the range $0.3 \leq d_2 \prec 1$ the predator species faces extinction and the system approaches asymptotically to the predator free equilibrium point E_2
6. Decreasing the values of predation rate P_2 in the range $P_2 \leq 0.34$ causes extinction in predator species and the system (1) approaches asymptotically to the predator free equilibrium point E_2.
7. Decreasing the values coefficient e_2 in the range $e_2 \leq 0.25$ has the same effects as that of P_2.
8. Finally it is observed that, varying each of the values of parameters K, C, n, P_1, e_1, d_1 has no effect on the dynamical behavior of the system (1) and the system still approaches asymptotically to coexistence equilibrium point.

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