Journal of Kerbala University , Vol. 13 No.2 Scientific . 2015

On fuzzy fractional differential equations using Riemann-Liouville derivative

حول معادلات تفاضلية ضبابية كسورية باستخدام مشتقة ريمان- ليوفيل

Ruaa Hameed Hassan and Amal K.Haydar
Department of Mathematics, Faculty of Education for Girls, Kufa University

Abstract

The main aim of this paper is to find the formulas of fuzzy fractional derivatives of order $0<\beta<2$ and the formulas of fuzzy Laplace transforms for fuzzy fractional derivatives of order $1<\beta<2$ using Riemann-Liouville derivative and H-differentiability.

المستخلص

الهدف الرئيسي من هذا البحث إيجاد صيغ المشتقات الكسورية الضبابية من الرتبة $0<\beta<2$، وصيغ تحويلات لابلاس الضبابية للمشتقات الكسورية الضبابية من الرتبة $1<\beta<2$ باستخدام مشتقة ريمان- ليوفيل وقابلية الاشتقاق-H.

1. Introduction

Fractional calculus theory is a mathematical analysis tool applied to the study of integrals and derivatives of arbitrary order which unifies and generalizes the notions of integer of-order differentiation of n -fold integration [1-3].

There are many works in subject of fractional calculus, recently, Salahshour et al. [4] deal with the solution of fuzzy fractional differential equations under Riemann-Liouville H-differentiability by fuzzy Laplace transforms. Ahmad et al. [5] deal with fuzzy power series which is a generalization to the classical power series. Allahviranloo et al. [6] give the explicit solutions of uncertain fractional differential equations under Riemann-Liouville and H-differentiability. This paper is organized as follows: Section 2 contains basic concepts. In section 3, we find the formulas of fuzzy Riemann-Liouville fractional derivatives of order $0<\beta<2$ and fuzzy Laplace transforms for fuzzy fractional Riemann-Liouville derivatives of order $1<\beta<2$. Also, an example is solved. Finally conclusions are drawn in section 4.

2.Basic Concepts

Definition 2.1 [1] The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ x>0.$$

Definition 2.2 [7] A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of function $\underline{u}(r)$ and $\overline{u}(r), 0\le r\le 1$ which satisfy the following requirements :

1. $\underline{u}(r)$ is bounded non – decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(r)$ is bounded non – increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(r)\le\overline{u}(r)$.

Definition 2.3 [7] Let $x,y\in E$. If there exists $z\in E$ such that $x=y+z$, then z is called H-difference of x and y, and it is denoted by $x\ominus y$.

We note that $x\ominus y\ne x+(-1)y$ and E be the set of all fuzzy numbers on R.

Definition 2.4 [8] Let $f(x)$ be continuous fuzzy-valued function. Suppose that $f(x)e^{-sx}$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^{\infty} f(x)e^{-sx}dx$ is called fuzzy Laplace transforms and is denoted as:

$$L[f(x)] = \int_0^{\infty} f(x)e^{-sx}dx \quad (s>0 \text{ and integer}).$$

We have:

$$\int_0^{\infty} f(x)e^{-sx}dx = \left[\int_0^{\infty} \underline{f}(x;r)e^{-sx}dx, \int_0^{\infty} \overline{f}(x;r)e^{-sx}dx\right],$$

also by using the definition of classical Laplace transform:

$$\ell[\underline{f}(x;r)] = \int_0^{\infty} \underline{f}(x;r)e^{-sx}dx \text{ and } \ell[\overline{f}(x;r)] = \int_0^{\infty} \overline{f}(x;r)e^{-sx}dx,$$

then we follow:

$$L[f(x)] = [\ell[\underline{f}(x;r)], \ell[\overline{f}(x;r)]].$$

Definition 2.5 [5] A real function $f(x), x>0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p>\mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x) \in C_\mu, n \in N$.

Definition 2.6 [5] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of function $f(x) \in C_\mu, \mu \geq -1$ is defined as :

$$J_s^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_s^x (x-t)^{\alpha-1} f(t)dt, & x>t>s\geq 0, \alpha>0, \\ f(x), & \alpha = 0. \end{cases}$$

Definition 2.7 [5] The Riemann-Liouville fractional derivative of order $\alpha>0$ of $f \in C_{-1}^n, n \in N$ is defined as:

$$D_s^{\alpha} f(x) = \begin{cases} \dfrac{d^n}{dx^n} J^{n-\alpha} f(x), & n-1<\alpha<n, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \end{cases}$$

Now, we denote $C^F[a,b]$ as the space of all continuous fuzzy-valued functions on $[a,b]$. Also we denote $L^F[a,b]$ as the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a,b] \subset R$. [4]

Definition 2.8 [4] Let $f(x) \in C^F[a,b] \cap L^F[a,b], x_0 \text{ in} (a,b)$ and $\phi(x) = \dfrac{1}{\Gamma[1-\beta]} \int_a^x \dfrac{f(t)dt}{(x-t)^{\beta}}$.

We say that $f(x)$ is fuzzy Riemann-Liouville H- differentiable about order $0<\beta<1$ at x_0, if there exists an element $\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) \in E$, such that for $h>0$, sufficiently small

(i) $$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0+h) \ominus \phi(x_0)}{h} = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0-h)}{h}$$

or

(ii) $$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0+h)}{-h} = \lim_{h\to 0+} \frac{\phi(x_0-h) \ominus \phi(x_0)}{-h}$$

(iii) $\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0+h) \ominus \phi(x_0)}{h} = \lim_{h\to 0+} \frac{\phi(x_0-h) \ominus \phi(x_0)}{-h}$

or

(iv) $\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0+h)}{-h} = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0-h)}{h}$

For the sake of simplicity , we say that the fuzzy-valued function f is ${}^{RL}\left[(i)-\beta\right]$ -differentiable if it is differentiable as in the definition 2.8 case (i), and f is ${}^{RL}\left[(ii)-\beta\right]$-differentiable if it is differentiable as in the definition 2.8 case (ii) and so on for the other cases.

Theorem 2.9 [4] Let $f(x) \in C^F[a,b] \cap L^F[a,b]$, x_0 in (a,b) and $0<\beta<1$. Then:

(i) Let us consider f is ${}^{RL}\left[(i)-\beta\right]$ - differentiable fuzzy-valued function, then:

$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \left[({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r) , ({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r)\right],\ 0 \le r \le 1$

(ii) Let us consider f is ${}^{RL}\left[(ii)-\beta\right]$ - differentiable fuzzy -valued function, then:

$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \left[({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r) , ({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r)\right],\ 0 \le \text{r} \le 1$

where

$$({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}$$

and

$$({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\overline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}$$

Theorem 2.10 [4] [Derivative theorem] suppose that $f(x) \in C^{\text{F}}[0,\infty) \cap L^{\text{F}}[0,\infty)$. Then :

$$L\left[\left({}^{RL}D_{a^+}^{\beta} f\right)(x)\right] = s^{\beta} L\left[f(x)\right] \ominus \left({}^{RL}D_{a^+}^{\beta-1} f\right)(0),$$

if f is ${}^{RL}\left[(i)-\beta\right]$ -differentiable, and

$$L\left[\left({}^{RL}D_{a^+}^{\beta} f\right)(x)\right] = -\left({}^{RL}D_{a^+}^{\beta-1} f\right)(0) \ominus \left(-s^{\beta} L\left[f(x)\right]\right),$$

if f is ${}^{RL}\left[(ii)-\beta\right]$-differentiable, provided the mentioned Hukuhara difference exist.

3. Riemann-Liouville H-differentiability

In this section, we introduce definition of fuzzy Riemann-Liouville derivatives of order $0<\beta<2$ under H-differentiability and find fuzzy Laplace transforms for fuzzy fractional derivatives of order $1<\beta<2$.

Definition 3.1 Let $f(x) \in C^F[0,b] \cap L^F[0,b]$, $\phi(x) = \frac{1}{\Gamma(\lceil\beta\rceil-\beta)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\lceil\beta\rceil+\beta}}$ where $\phi_1(x_0)$ and $\phi_2(x_0)$ are the limits defined in a_1 and a_2 respectively. $f(x)$ is the Riemann-Liouville type fuzzy fractional differentiable function of order $0<\beta<2$, $\beta\neq 1$ at $x_0\in(0,b)$, if there exists an element $\left({}^{RL}D^{\beta}f\right)(x_0)\in C^F$ such that for all $0\le r\le 1$ and for $h>0$ sufficiently near zero either:

a$_1$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi(x_0+h)\ominus\phi(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{\phi(x_0)\ominus\phi(x_0-h)}{h}$ or

a$_2$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi(x_0)\ominus\phi(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{\phi(x_0-h)\ominus\phi(x_0)}{-h}$

for $0<\beta<1$ and either:

b$_1$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi_1(x_0+h)\ominus\phi_1(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{\phi_1(x_0)\ominus\phi_1(x_0-h)}{h}$ or

b$_2$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi_1(x_0)\ominus\phi_1(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{\phi_1(x_0-h)\ominus\phi_1(x_0)}{-h}$ or

b$_3$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi_2(x_0+h)\ominus\phi_2(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{\phi_2(x_0)\ominus\phi_2(x_0-h)}{h}$ or

b$_4$. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{\phi_2(x_0)\ominus\phi_2(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{\phi_2(x_0-h)\ominus\phi_2(x_0)}{-h}$

for $1<\beta<2$.

If the fuzzy valued function $f(x)$ is differentiable as in definition 3.1 cases (a$_1$, b$_1$, b$_3$) it is the Riemann-Liouville type differentiable in the first form and denoted by $\left({}^{RL}D_1^{\beta}f\right)(x_0)$, $\left({}^{RL}D_{1,1}^{\beta}f\right)(x_0)$ and $\left({}^{RL}D_{2,1}^{\beta}f\right)(x_0)$ respectively. If the fuzzy valued function $f(x)$ is differentiable as in definition 3.1 cases (a$_2$, b$_2$, b$_4$) it is the Riemann-Liouville type differentiable in the second form and denoted by $\left({}^{RL}D_2^{\beta}f\right)(x_0)$, $\left({}^{RL}D_{1,2}^{\beta}f\right)(x_0)$ and $\left({}^{RL}D_{2,2}^{\beta}f\right)(x_0)$ respectively.

Theorem 3.2 Let $f(x)\in C^F[0,b]\cap L^F[0,b]$ be a fuzzy-valued function and $f(x)=\left[\underline{f}(x;r),\overline{f}(x;r)\right]$ for $r\in[0,1]$, $0<\beta<2$ and $x_0\in(0,b)$. Then:

a$_1$. If $f(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $0<\beta<1$

$$\left({}^{RL}D_1^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

a$_2$. If $f(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $0<\beta<1$

$$\left({}^{RL}D_2^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b$_1$. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{RL}D_{1,1}^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

b$_2$. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{RL}D_{1,2}^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b$_3$. If $\left({}^{RL}D_2^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{RL}D_{2,1}^{\beta} f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b4. If $\left({}^{RL}D_2^{\beta} f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{RL}D_{2,2}^{\beta} f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r)\right]$$

where

$$\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\left(\frac{d}{dx}\right)^{\lceil\beta\rceil}\int_0^x\frac{\underline{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0}$$

$$\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\left(\frac{d}{dx}\right)^{\lceil\beta\rceil}\int_0^x\frac{\bar{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0}$$

Proof We shall prove b1 as follows: Since $\left({}^{RL}D_1^{\beta} f\right)(x), 1<\beta<2$ is the Riemann-Liouville type fuzzy fractional differentiable function in the first form, then from b1, of definition 3.1, we have:

$$\phi_1(x_0+h)\ominus\phi_1(x_0)=\left[\underline{\phi}_1(x_0+h;r)-\underline{\phi}_1(x_0;r),\bar{\phi}_1(x_0+h;r)-\bar{\phi}_1(x_0;r)\right],$$
$$\phi_1(x_0)\ominus\phi_1(x_0-h)=\left[\underline{\phi}_1(x_0;r)-\underline{\phi}_1(x_0-h;r),\bar{\phi}_1(x_0;r)-\bar{\phi}_1(x_0-h;r)\right].$$

Multiplying both sides by $\frac{1}{h}$, $h>0$, we get:

$$\frac{\phi_1(x_0+h)\ominus\phi_1(x_0)}{h}=\left[\frac{\underline{\phi}_1(x_0+h;r)-\underline{\phi}_1(x_0;r)}{h},\frac{\bar{\phi}_1(x_0+h;r)-\bar{\phi}_1(x_0;r)}{h}\right],$$
$$\frac{\phi_1(x_0)\ominus\phi_1(x_0-h)}{h}=\left[\frac{\underline{\phi}_1(x_0;r)-\underline{\phi}_1(x_0-h;r)}{h},\frac{\bar{\phi}_1(x_0;r)-\bar{\phi}_1(x_0-h;r)}{h}\right].$$

By taking $h\to 0^+$ on both sides of the above relation, we get:

$$\left({}^{RL}D^{\beta} f\right)(x_0)=\left[\frac{d}{dx}\underline{\phi}_1(x_0;r),\frac{d}{dx}\bar{\phi}_1(x_0;r)\right] \tag{3.1}$$

Now, since $\phi_1(x_0)$ is equal to the limits defined in a1 of definition 3.1, then we have:

$$\phi(x_0+h)\ominus\phi(x_0)=\left[\underline{\phi}(x_0+h;r)-\underline{\phi}(x_0;r),\bar{\phi}(x_0+h;r)-\bar{\phi}(x_0;r)\right],$$
$$\phi(x_0)\ominus\phi(x_0-h)=\left[\underline{\phi}(x_0;r)-\underline{\phi}(x_0-h;r),\bar{\phi}(x_0;r)-\bar{\phi}(x_0-h;r)\right].$$

Multiplying both sides by $\frac{1}{h}$, $h>0$, we obtain

$$\frac{\phi(x_0+h)\ominus\phi(x_0)}{h}=\left[\frac{\underline{\phi}(x_0+h;r)-\underline{\phi}(x_0;r)}{h},\frac{\bar{\phi}(x_0+h;r)-\bar{\phi}(x_0;r)}{h}\right],$$
$$\frac{\phi(x_0)\ominus\phi(x_0-h)}{h}=\left[\frac{\underline{\phi}(x_0;r)-\underline{\phi}(x_0-h;r)}{h},\frac{\bar{\phi}(x_0;r)-\bar{\phi}(x_0-h;r)}{h}\right].$$

By taking $h\to 0^+$ on both sides of the above relation ,we get:

$$\phi_1(x_0)=\left[\underline{\phi}'(x_0;r),\bar{\phi}'(x_0;r)\right],$$

Then

$$\underline{\phi}_1(x_0;r)=\underline{\phi}'(x_0;r)\ ,\ \bar{\phi}_1(x_0;r)=\bar{\phi}'(x_0;r) \tag{3.2}$$

Substituting (3.2) in (3.1) yields:

$$\left({}^{RL}D^{\beta} f\right)(x_0)=\left[\frac{d^2}{dx^2}\underline{\phi}\,(x_0;r),\frac{d^2}{dx^2}\overline{\phi}\,(x_0;r)\right]$$
$$=\left[\left({}^{RL}D^{\beta}\, \underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\, \overline{f}\right)(x_0;r)\right].$$

The other proofs are similar .

Theorem 3.3 Suppose that $f\in C^{F}[0,\infty)\bigcap L^{F}[0,\infty)$ and, $1<\beta<2$ then :

1. If $\left({}^{RL}D_1^{\beta} f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable fuzzy–valued function, then
$L\left[\left({}^{RL}D_{1,1}^{\beta} f\right)(x)\right]=s^{\beta}L[f(x)]\ominus s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left({}^{RL}D^{\beta-1} f\right)(0)$
2. If $\left({}^{RL}D_1^{\beta} f\right)(x)$ is ${}^{RL}[(ii)-\beta]$-differentiable fuzzy–valued function, then
$L\left[\left({}^{RL}D_{1,2}^{\beta} f\right)(x)\right]=-s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left(-s^{\beta}\right)L[f(x)]-\left({}^{RL}D^{\beta-1} f\right)(0)$
3. If $\left({}^{RL}D_2^{\beta} f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable fuzzy–valued function, then
$L\left[\left({}^{RL}D_{2,1}^{\beta} f\right)(x)\right]=-s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left(-s^{\beta}\right)L[f(x)]\ominus\left({}^{RL}D^{\beta-1} f\right)(0)$
4. If $\left({}^{RL}D_2^{\beta} f\right)(x)$ is ${}^{RL}[(ii)-\beta]$-differentiable fuzzy–valued function, then
$L\left[\left({}^{RL}D_{2,2}^{\beta} f\right)(x)\right]=s^{\beta}L[f(x)]\ominus s\left({}^{RL}D^{\beta-2} f\right)(0)-\left({}^{RL}D^{\beta-1} f\right)(0)$

Proof To prove **3,** let us consider $\left({}^{RL}D_2^{\beta} f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable, then by theorem 3.2 we get:

$$L\left[\left({}^{RL}D_{2,1}^{\beta} f\right)(x)\right]=L\left[\left(\underline{{}^{RL}D^{\beta} f}\right)(x;r),\left(\overline{{}^{RL}D^{\beta} f}\right)(x;r)\right]$$
$$=\left[\ell\left[\left({}^{RL}D^{\beta}\,\overline{f}\right)(x;r)\right],\ell\left[\left({}^{RL}D^{\beta}\,\underline{f}\right)(x;r)\right]\right] \qquad (3.3)$$

We know that Laplace transform of Riemann-Liouville fractional derivative of order $1<\beta<2$ is:

$$\ell\left({}^{RL}D^{\beta}\underline{f}\right)(x;r)=s^{\beta}\ell\left[\underline{f}(x;r)\right]-\left({}^{RL}D^{\beta-1}\underline{f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\underline{f}\right)(0;r) \qquad (3.4)$$
$$\ell\left({}^{RL}D^{\beta}\overline{f}\right)(x;r)=s^{\beta}\ell\left[\overline{f}(x;r)\right]-\left({}^{RL}D^{\beta-1}\overline{f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\overline{f}\right)(0;r) \qquad (3.5)$$

Since $0<\beta-1<1$, then we have:

$$\left({}^{RL}D^{\beta-1}\underline{f}\right)(0;r)=\left(\overline{{}^{RL}D^{\beta-1}f}\right)(0;r),\ \left({}^{RL}D^{\beta-1}\overline{f}\right)(0;r)=\left(\underline{{}^{RL}D^{\beta-1}f}\right)(0;r). \qquad (3.6)$$

Using the equations (3.4)-(3.6), then equation (3.3) becomes:

$$L\left[\left({}^{RL}D^{\beta} f\right)(x)\right]=\left[s^{\beta}\ell\left[\overline{f}(x;r)\right]-\left(\underline{{}^{RL}D^{\beta-1}f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\overline{f}\right)(0;r),s^{\beta}\ell\left[\underline{f}(x;r)\right]-\right.$$

$$\left.\left(\overline{{}^{RL}D^{\beta-1}f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\underline{f}\right)(0;r)\right]$$
$$=-s\left({}^{RL}D^{\beta-2}f\right)(0)\ominus\left(-s^{\beta}\right)L[f(x)]\ominus\left({}^{RL}D^{\beta-1}f\right)(0).$$

The other proofs are similar .

Example 3.4 Consider the following FFIVP:

$$\left({}^{RL}D^{\beta}y\right)(x)=\lambda y(x)\ ,\ \lambda\in(0,\infty),1<\beta<2 \qquad (3.7)$$

$$\left({}^{RL}D^{\beta-1}y\right)(0)={}^{RL}y_0^{(\beta-1)}\in E\ ,$$

$$\left({}^{RL}D^{\beta-2}y\right)(0)={}^{RL}y_0^{(\beta-2)}\in E.$$

We note that:

$$\left(\underline{{}^{RL}D^{\beta-1}y}\right)(0;r)={}^{RL}\underline{y}_0^{(\beta-1)}(r),\left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r)={}^{RL}\overline{y}_0^{(\beta-1)}(r),$$

$$\left({}^{RL}D^{\beta-2}\underline{y}\right)(0;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r),\left({}^{RL}D^{\beta-2}\overline{y}\right)(0;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r).$$

By taking fuzzy Laplace transform for both sides of equation (3.7) we get:

$$L\left[\left({}^{RL}D^{\beta}y\right)(x)\right]=\lambda L\left[y(x)\right]. \qquad (3.8)$$

Now, we have $2^2=4$ cases as follows:

Case 1 Let us consider $\left({}^{RL}D_1^{\beta}y\right)(x)$ be ${}^{RL}[i-\beta]$-differentiable fuzzy-valued function, then equation (3.8) can be written as follows:

$$s^{\beta}L\left[y(x)\right] \ominus s\left({}^{RL}D^{\beta-2}y\right)(0)\ominus \left({}^{RL}D^{\beta-1}y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\underline{y}\right)(0;r)-\left(\underline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\overline{y}\right)(0;r)-\left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Therefore we have :

$$\left(s^{\beta}-\lambda\right)\ell\left[\underline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r),$$
$$\left(s^{\beta}-\lambda\right)\ell\left[\overline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r).$$

Consequently , applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right],$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right].$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

where $E_{\alpha,\beta}(z)$ denotes the Mittag-Leffler function .

Case 2 Let us consider $\left({}^{RL}D_1^{\beta}y\right)(x)$ be ${}^{RL}[ii-\beta]$-differentiable fuzzy-valued functions, then equation (3.8) can be written as follows:

$$-s\left({}^{RL}D^{\beta-2}y\right)(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]-\left({}^{RL}D^{\beta-1}y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$-s\left({}^{RL}D^{\beta-2}\,\overline{y}\right)(0;r)+s^{\beta}\ell\left[\overline{y}(x;r)\right]-\left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-\left(\underline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Then we get the system:

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-\lambda\ell\left[\underline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r),$$
$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-\lambda\ell\left[\overline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r).$$

The solution of the above system is:

$$\ell\left[\underline{y}(x;r)\right]=\frac{{}^{RL}\underline{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\underline{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\ {}^{RL}\overline{y}_0^{(\beta-2)}(r)s+\lambda\ {}^{RL}\overline{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2},$$
$$\ell\left[\overline{y}(x;r)\right]=\frac{{}^{RL}\overline{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\overline{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\ {}^{RL}\underline{y}_0^{(\beta-2)}(r)s+\lambda\ {}^{RL}\underline{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2}.$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\ {}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\ {}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\ {}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\ {}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\ {}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\ {}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\ {}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\ {}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right).$$

Case 3 Let us consider that $\left({}^{RL}D_2^{\beta}y\right)(x)$ be ${}^{RL}\left[i-\beta\right]$-differentiable fuzzy-valued function then equation (3.8) can be written as follows:

$$-s\left({}^{RL}D^{\beta-2}y\right)(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]\ominus\left({}^{RL}D^{\beta-1}y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$-s\left({}^{RL}D^{\beta-2}\,\overline{y}\right)(0;r)+s^{\beta}\ell\left[\overline{y}\left(x;r\right)\right]-\left(\underline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\underline{y}\left(x;r\right)\right],$$

$$-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)+s^{\beta}\ell\left[\underline{y}\left(x;r\right)\right]-\left(\overline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\overline{y}\left(x;r\right)\right].$$

Then we get the system:

$$s^{\beta}\ell\left[\overline{y}\left(x;r\right)\right]-\lambda\ell\left[\underline{y}\left(x;r\right)\right]={}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)s+{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right),$$
$$s^{\beta}\ell\left[\underline{y}\left(x;r\right)\right]-\lambda\ell\left[\overline{y}\left(x;r\right)\right]={}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)s+{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right).$$

The solution of the above system is

$$\ell\left[\underline{y}\left(x;r\right)\right]=\frac{{}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)s^{\beta+1}+{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)s^{\beta}+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)s+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)}{s^{2\beta}-\lambda^{2}}$$
$$\ell\left[\overline{y}\left(x;r\right)\right]=\frac{{}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)s^{\beta+1}+{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)s^{\beta}+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)s+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)}{s^{2\beta}-\lambda^{2}}$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}\left(x;r\right)={}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^{2}}\right]+{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^{2}}\right]+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^{2}}\right]$$
$$+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^{2}}\right]$$
$$\overline{y}\left(x;r\right)={}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^{2}}\right]+{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^{2}}\right]+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^{2}}\right]$$
$$+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^{2}}\right]$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}\left(x;r\right)={}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^{2}x^{2\beta}\right)+{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^{2}x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^{2}x^{2\beta}\right)+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^{2}x^{2\beta}\right),$$

$$\overline{y}\left(x;r\right)={}^{RL}\overline{y}_{0}^{(\beta-2)}\left(r\right)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^{2}x^{2\beta}\right)+{}^{RL}\underline{y}_{0}^{(\beta-1)}\left(r\right)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^{2}x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\underline{y}_{0}^{(\beta-2)}\left(r\right)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^{2}x^{2\beta}\right)+\lambda\;{}^{RL}\overline{y}_{0}^{(\beta-1)}\left(r\right)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^{2}x^{2\beta}\right).$$

Case 4 Let us consider that $\left({}^{RL}D_{2}^{\beta}y\right)\left(x\right)$ be ${}^{RL}\left[ii-\beta\right]$-differentiable fuzzy-valued function then equation (3.8) can be written as follows:

$$s^{\beta}L\left[y\left(x\right)\right]\ominus s\left({}^{RL}D^{\beta-2}\,y\right)(0)-\left({}^{RL}D^{\beta-1}\,y\right)(0)=\lambda L\left[y\left(x\right)\right].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)-\left(\overline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\,\overline{y}\right)(0;r)-\left(\underline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Therefore we have :

$$\left(s^{\beta}-\lambda\right)\ell\left[\underline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r),$$
$$\left(s^{\beta}-\lambda\right)\ell\left[\overline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r).$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right],$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right].$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right).$$

4. Conclusions

In this paper the Riemann-Liouville fractional derivatives of order $0<\beta<2$ for fuzzy-valued function f are studied. Also the fuzzy Laplace transforms for the Riemann-Liouville fractional derivatives of order $1<\beta<2$ under H-differentiability are discussed. Also an fuzzy fractional initial value problem of order $1<\beta<2$ is solved.

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