Journal of Kerbala University , Vol. 13 No.2 Scientific . 2015

Generalization of Fuzzy Laplace Transforms for Fuzzy Derivatives

تعميم تحويلات لابلاس الضبابية للمشتقات الضبابية

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Abstract

The main aim of this paper is to find the general formula to the fuzzy derivative of the n-th order and the general formula for fuzzy Laplace transforms to the fuzzy derivative of the n-th order by using generalized H-differentiability.

المستخلص

الهدف الرئيسي من هذا البحث ايجاد الصيغة العامة للمشتقة الضبابية من الرتبة n وايضا الصيغة العامة الى تحويلات لابلاس الضبابية العامة للمشتقة الضبابية من الرتبة n باستخدام قابلية الاشتقاق $H-$ المعممة .

1. Introduction

The concept of the fuzzy derivative was first introduced by Chang and Zadeh in [1], it was followed up by Dubois and Prade in 1982 [2] and Puri and Ralescu in 1983 [3]. Abbasbandy et al. [4] studied a numerical method for n-th order fuzzy differential equations based on Seikkala derivative with initial value conditions, Allahviranloo et al. [5] considered eigenvalue-eigenvector method for solving n-th order fuzzy differential equations with fuzzy initial conditions, Bede et al. [6] provided solution of first order linear fuzzy differential equations by variation of constant formula. . Recently, many researchers worked on solving fuzzy differential equations by using fuzzy Laplace transforms, especially in [7,8,9].

This paper is arranged as follows: Basic concepts are given in Section 2. In Section 3, the general formula for the fuzzy derivative of the n-th order and fuzzy Laplace transform for the fuzzy derivative of the n-th order are found. In Section 4, an example of the third order is solved. In Section 5, conclusions are drawn.

2. Basic Concepts

In this section, some necessary definitions and concepts are introduced:

Definition 2.1 [8] A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha),\ 0\le\alpha\le1$ which satisfy the following requirements:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuousat 0,
2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuousat 0,
3. $\underline{u}(\alpha)\le\overline{u}(\alpha),\ 0\le\alpha\le1.$

Definition 2.2 [8] Let $x,y\in E$. If there exists $z\in E$ such that $x=y+z$, then z is called the H-difference of x and y, and it is denoted by $x\ominus y$. In this paper, the sign "$\ominus$" always stands for H-difference, and also note that $x\ominus y\neq x+(-1)y$.

Definition 2.3 [8] Let $f:(a,b)\to E$ and $x_0\in(a,b)$. We say that f is strongly generalized differential at x_0 if there exists an element $f'(x_0)\in E$, such that

i. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ \Theta\ f(x_0)$, $\exists f(x_0)\Theta f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\Theta f(x_0))/h]=\lim_{h\to 0}[(f(x_0)\Theta f(x_0-h))/h]=f'(x_0) \text{ or}$$

ii. For all $h>0$ sufficiently small, $\exists f(x_0)\Theta f(x_0+h)$, $\exists f(x_0-h)\Theta f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\Theta f(x_0+h))/(-h)]=\lim_{h\to 0}[(f(x_0-h)\Theta f(x_0))/(-h)]=f'(x_0) \text{ or}$$

iii. For all $h>0$ sufficiently small, $\exists f(x_0+h)\Theta f(x_0)$, $\exists f(x_0-h)\Theta f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\Theta f(x_0))/h]=\lim_{h\to 0}[(f(x_0-h)\Theta f(x_0))/(-h)]=f'(x_0) \text{ or}$$

iv. For all $h>0$ sufficiently small, $\exists f(x_0)\Theta f(x_0+h)$, $\exists f(x_0)\Theta f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\Theta f(x_0+h))/(-h)]=\lim_{h\to 0}[(f(x_0)\Theta f(x_0-h))/h]=f'(x_0)$$

Definition 2.4 [7] Let $f(t)$ be continuous fuzzy-valued function. Suppose that $f(t)\,e^{-st}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty f(t)\,e^{-st}dt$ is called fuzzy Laplace transforms and is denoted as $L(f(t))=\int_0^\infty f(t)\cdot e^{-st}dt,\ (s>0)$. We have

$$\int_0^\infty f(t)\,\mathrm{e}^{-st}dt=(\int_0^\infty \underline{f}(t,\alpha)\,\mathrm{e}^{-st}dt,\ \int_0^\infty \overline{f}(t,\alpha)\,\mathrm{e}^{-st}dt),$$

also by using the definition of classical Laplace transform

$$l[\underline{f}(t,\alpha)]=\int_0^\infty \underline{f}(t,\alpha)\cdot e^{-st}dt \text{ and } l[\overline{f}(t,\alpha)]=\int_0^\infty \overline{f}(t,\alpha)\cdot e^{-st}dt,$$

then, we follow:

$$L(f(t))=(l(\underline{f}(t,\alpha)),l(\overline{f}(t,\alpha))).$$

3. Fuzzy Laplace Transforms for the *n*-th Derivative

In this section, we find a general form for the fuzzy derivative of any order n, $n\in Z^+$. Also, we find, a generalization for fuzzy Laplace transforms for n-th derivative.

Theorem 3.1 Suppose that $F(t),F'(t),\cdots,F^{(n-1)}(t)$ are differentiable fuzzy valued functions such that $F^{(i_1)}(t),F^{(i_2)}(t),\cdots,F^{(i_m)}(t)$ are (ii)-differentiable functions for $0\le i_1<i_2<\ldots<i_m\le n-1$, $0\le m\le n$ and $F^{(p)}(t)$ is (i)-differentiable for $p\ne i_j$, $j=1,2,\ldots,m$, and if α-cut representation of fuzzy- valued function $F(t)$ is denoted by $[F(t)]^\alpha=[f_\alpha(t),g_\alpha(t)]$, then:

(a) If m is an even number then $[F^{(n)}(t)]^\alpha=[f_\alpha^{(n)}(t),g_\alpha^{(n)}(t)]$.

(b) If m is an odd number then $[F^{(n)}(t)]^\alpha=[g_\alpha^{(n)}(t),f_\alpha^{(n)}(t)]$.

Proof We shall prove by mathematical induction on n. We have the relations (a) and (b) are true for $n=1$ [7].

We suppose that the relations (a) and (b) are true for $n = k$, i.e., if $F, F', \cdots, F^{(k-1)}$ are differentiable fuzzy-valued functions such that $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ are (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le k-1$, $0 \le m \le k$, then

If m is an even number we have:

$$[F^{(k)}(t)]^\alpha = [f_\alpha^{(k)}(t), g_\alpha^{(k)}(t)], \qquad (3.1)$$

and if m is an odd number we have:

$$[F^{(k)}(t)]^\alpha = [g_\alpha^{(k)}(t), f_\alpha^{(k)}(t)]. \qquad (3.2)$$

Now, we prove that the relations (a) and (b) are true for $n = k+1$ as follows:

(a) Let $F, F', \cdots, F^{(k)}$ be differentiable fuzzy valued functions such that $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ are (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le k$, $0 \le m \le k+1$ and m is an even number. If $m = 0$ then there is no (ii)-differentiable functions, i.e., $F(t), F'(t), \cdots, F^{(k)}(t)$ are (i)-differentiable. Since $F(t), F'(t), \cdots, F^{(k-1)}(t)$ are (i)-differentiable then by relation (3.1) of induction hypothesis we have:

$$[F^{(k)}(t)]^\alpha = [f_\alpha^{(k)}(t), g_\alpha^{(k)}(t)].$$

Since $F^{(k)}$ is (i)-differentiable then from definition 2.3, we have:

$$[F^{(k)}(t+h) \Theta F^{(k)}(t)]^\alpha = = [f_\alpha^{(k)}(t+h) - f_\alpha^{(k)}(t), g_\alpha^{(k)}(t+h) - g_\alpha^{(k)}(t)],$$

$$[F^{(k)}(t) \Theta F^{(k)}(t-h)]^\alpha = [f_\alpha^{(k)}(t) - f_\alpha^{(k)}(t-h), g_\alpha^{(k)}(t) - g_\alpha^{(k)}(t-h)].$$

and, multiplying by $\frac{1}{h}, h > 0$ we get:

$$\frac{1}{h}[F^{(k)}(t+h) \Theta F^{(k)}(t)]^\alpha = [\frac{f_\alpha^{(k)}(t+h) - f_\alpha^{(k)}(t)}{h}, \frac{g_\alpha^{(k)}(t+h) - g_\alpha^{(k)}(t)}{h}],$$

and

$$\frac{1}{h}[F^{(k)}(t) \Theta F^{(k)}(t-h)]^\alpha = [\frac{f_\alpha^{(k)}(t) - f_\alpha^{(k)}(t-h)}{h}, \frac{g_\alpha^{(k)}(t) - g_\alpha^{(k)}(t-h)}{h}].$$

Using $h \longrightarrow 0$ on both sides of aforementioned relation, we get

$$[F^{(k+1)}(t)]^\alpha = [f_\alpha^{(k+1)}(t), g_\alpha^{(k+1)}(t)]. \qquad (3.3)$$

Then the relation (a) is true if $m = 0$.

Now, if m is an even number such that $2 \le m \le k+1$, then we have two possibilities for the functions $F, F', ..., F^{(k-1)}$ as follows:

Either $F, F', \cdots, F^{(k)}$ contain an even number m and $F, F', \cdots, F^{(k-1)}$ contain an even number m of (ii)-differentiable functions $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ then $F^{(k)}$ is (i)-differentiable. By relation (3.1) of induction hypothesis, we have $[F^{(k)}(t)]^\alpha = [f_\alpha^{(k)}(t), g_\alpha^{(k)}(t)]$.

Since $F^{(k)}$ is (i)-differentiable then from definition 2.3, we have:

$$[F^{(k)}(t+h) \Theta F^{(k)}(t)]^\alpha = = [f_\alpha^{(k)}(t+h) - f_\alpha^{(k)}(t), g_\alpha^{(k)}(t+h) - g_\alpha^{(k)}(t)],$$

$$[F^{(k)}(t) \Theta F^{(k)}(t-h)]^\alpha = [f_\alpha^{(k)}(t) - f_\alpha^{(k)}(t-h), g_\alpha^{(k)}(t) - g_\alpha^{(k)}(t-h)].$$

and, multiplying by $\frac{1}{h}, h > 0$ we get:

$$\frac{1}{h}[F^{(k)}(t+h) \Theta F^{(k)}(t)]^\alpha = [\frac{f_\alpha^{(k)}(t+h) - f_\alpha^{(k)}(t)}{h}, \frac{g_\alpha^{(k)}(t+h) - g_\alpha^{(k)}(t)}{h}],$$

and

$$\frac{1}{h}[F^{(k)}(t)\ \Theta\ F^{(k)}(t-h)]^{\alpha}=[\frac{f_{\alpha}{}^{(k)}(t)-f_{\alpha}{}^{(k)}(t-h)}{h},\frac{g_{\alpha}{}^{(k)}(t)-g_{\alpha}{}^{(k)}(t-h)}{h}].$$

Using $h \longrightarrow 0$ on both sides of aforementioned relation, we get the relation (3.3).

Or $F, F', \cdots, F^{(k)}$ contain an even number m and $F, F', \cdots, F^{(k-1)}$ contain an odd number $(m-1)$ of (ii)-differentiable functions $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ then $F^{(k)}$ is (ii)-differentiable. By (3.2) of induction hypothesis, we have:

$$[F^{(k)}(t)]^{\alpha}=[g_{\alpha}{}^{(k)}(t), f_{\alpha}{}^{(k)}(t)].$$

Since $F^{(k)}(t)$ is (ii)-differentiable then from definition 2.3, we have:

$$[F^{(k)}(t)\Theta F^{(k)}(t+h)]^{\alpha}=[g_{\alpha}{}^{(k)}(t)-g_{\alpha}{}^{(k)}(t+h), f_{\alpha}{}^{(k)}(t)-f_{\alpha}{}^{(k)}(t+h)],$$

$$[F^{(k)}(t-h)\Theta F^{(k)}(t)]^{\alpha}=[g_{\alpha}{}^{(k)}(t-h)-g_{\alpha}{}^{(k)}(t), f_{\alpha}{}^{(k)}(t-h)-f_{\alpha}{}^{(k)}(t)].$$

and , multiplying by $\frac{1}{-h}$, $h>0$ we get:

$$\frac{1}{-h}[F^{(k)}(t)\Theta F^{(k)}(t+h)]^{\alpha}=[\frac{f_{\alpha}{}^{(k)}(t+h)-f_{\alpha}{}^{(k)}(t)}{h},\frac{g_{\alpha}{}^{(k)}(t+h)-g_{\alpha}{}^{(k)}(t)}{h}],$$

and

$$\frac{1}{-h}[F^{(k)}(t-h)\ \Theta\ F^{(k)}(t)]^{\alpha}=[\frac{f_{\alpha}{}^{(k)}(t)-f_{\alpha}{}^{(k)}(t-h)}{h},\frac{g_{\alpha}{}^{(k)}(t)-g_{\alpha}{}^{(k)}(t-h)}{h}].$$

Using $h \longrightarrow 0$ on both sides of aforementioned relation, we get the relation (3.3). Then relation (a) is true if m is an even number such that $2 \le m \le k+1$. Then (a) is true if $n=k+1$ and m is an even number such that $0 \le m \le k+1$.

Then relation (a) is true for any positive integer n.

(b) Let $F, F', \cdots, F^{(k)}$ be differentiable fuzzy-valued functions such that $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ are (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots \le k$, $1 \le m \le k+1$ and m is an odd number. Also, we have two possibilities for the functions $F, F', \cdots, F^{(k-1)}$ as follows:

Either $F, F', \cdots, F^{(k)}$ contain an odd number m and $F, F', \cdots, F^{(k-1)}$ contain an even number $(m-1)$ of (ii)-differentiable functions $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ then $F^{(k)}$ is (ii)-differentiable. By relation (3.1), we get

$$[F^{(k)}(t)]^{\alpha}=[f_{\alpha}{}^{(k)}(t), g_{\alpha}{}^{(k)}(t)].$$

Since $F^{(k)}$ is (ii)-differentiable then from definition 2.3, we have:

$$[F^{(k)}(t)\Theta F^{(k)}(t+h)]^{\alpha}=[f_{\alpha}{}^{(k)}(t)-f_{\alpha}{}^{(k)}(t+h), g_{\alpha}{}^{(k)}(t)-g_{\alpha}{}^{(k)}(t+h)],$$

$$[F^{(k)}(t-h)\Theta F^{(k)}(t)]^{\alpha}=[f_{\alpha}{}^{(k)}(t-h)-f_{\alpha}{}^{(k)}(t), g_{\alpha}{}^{(k)}(t-h)-g_{\alpha}{}^{(k)}(t)].$$

and , multiplying by $\frac{1}{-h}$, $\mathrm{h}>0$ we get:

$$\frac{1}{-h}[F^{(k)}(t)\Theta F^{(k)}(t+h)]^{\alpha}=[\frac{g_{\alpha}{}^{(k)}(t+h)-g_{\alpha}{}^{(k)}(t)}{h},\frac{f_{\alpha}{}^{(k)}(t+h)-f_{\alpha}{}^{(k)}(t)}{h}],$$

and

$$\frac{1}{-h}[F^{(k)}(t-h)\ \Theta\ F^{(k)}(t)]^{\alpha}=[\frac{g_{\alpha}{}^{(k)}(t)-g_{\alpha}{}^{(k)}(t-h)}{h},\frac{f_{\alpha}{}^{(k)}(t)-f_{\alpha}{}^{(k)}(t-h)}{h}].$$

Using $h \longrightarrow 0$ on both sides of aforementioned relation, we get

$$[F^{(k+1)}(t)]^{\alpha} = [g_{\alpha}{}^{(k+1)}(t), f_{\alpha}{}^{(k+1)}(t)] \qquad (3.4)$$

Or $F, F', \cdots, F^{(k)}$ contain an odd number m and $F, F', \cdots, F^{(k-1)}$ contain an odd number m of (ii)-differentiable functions $F^{(i_1)}, F^{(i_2)}, \cdots, F^{(i_m)}$ then $F^{(k)}$ is (i)-differentiable. By relation (3. 2), we get:

$$[F^{(k)}(t)]^{\alpha} = [g_{\alpha}{}^{(k)}(t), f_{\alpha}{}^{(k)}(t)].$$

Since $F^{(k)}(t)$ is (i)-differentiable then from definition 2.3, we have:

$$[F^{(k)}(t+h) \ominus F^{(k)}(t)]^{\alpha} = [g_{\alpha}{}^{(k)}(t+h) - g_{\alpha}{}^{(k)}(t), f_{\alpha}{}^{(k)}(t+h) - f_{\alpha}{}^{(k)}(t)],$$

$$[F^{(k)}(t) \ominus F^{(k)}(t-h)]^{\alpha} = [g_{\alpha}{}^{(k)}(t) - g_{\alpha}{}^{(k)}(t-h), f_{\alpha}{}^{(k)}(t) - f_{\alpha}{}^{(k)}(t-h)].$$

and , multiplying by $\frac{1}{h}$, $h > 0$ we get:

$$\frac{1}{h}[F^{(k)}(t+h) \ominus F^{(k)}(t)]^{\alpha} = [\frac{g_{\alpha}{}^{(k)}(t+h) - g_{\alpha}{}^{(k)}(t)}{h}, \frac{f_{\alpha}{}^{(k)}(t+h) - f_{\alpha}{}^{(k)}(t)}{h}],$$

and

$$\frac{1}{h}[F^{(k)}(t) \ominus F^{(k)}(t-h)]^{\alpha} = [\frac{g_{\alpha}{}^{(k)}(t) - g_{\alpha}{}^{(k)}(t-h)}{h}, \frac{f_{\alpha}{}^{(k)}(t) - f_{\alpha}{}^{(k)}(t-h)}{h}].$$

Finally, using $h \longrightarrow 0$ on both sides of aforementioned relation, we get the relation (3.4), then the relation (b) is true when $n = k+1$ and m is an odd such that $1 \le m \le k+1$.

Then the relation (b) is true for any positive integer n .

Thus, the theorem is true for any positive integer n .

Remark 3.2 If we put $n = 1, 2, 3, 4$ in theorem 3.1, we get the same results given in [7,8,9,9] respectively.

Theorem 3.3 Suppose that $g(t), g'(t), \ldots, g^{(n-1)}(t)$ be continuous fuzzy-valued functions on $[0, \infty)$ and of exponential order and that $g^{(n)}(t)$ is piecewise continuous fuzzy-valued function on $[0, \infty)$. Let $g^{(i_1)}(t), g^{(i_2)}(t), \ldots, g^{(i_m)}(t)$ be (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ and $g^{(p)}$ be (i)-differentiable function for $p \ne i_j$, $j = 1, 2, \ldots, m$ and $g(t) = (\underline{g}(t,\alpha), \overline{g}(t,\alpha))$;then

(1)If m is an even number, we have

$$L(g^{(n)}(t)) = s^n L(g(t)) \ominus s^{n-1} g(0) \otimes \sum_{k=1}^{n-1} s^{n-(k+1)} g^{(k)}(0), \qquad (3.5)$$

such that

$$\otimes = \begin{cases} \ominus, & \text{if the number of (ii)} - \text{differentiable functions } g^{(i)}, \text{provided that } i < k \text{ is an even number} \\ -, & \text{if the number of (ii)} - \text{differentiable functions } g^{(i)}, \text{provided that } i < k \text{ is an odd number} \end{cases} \qquad (3.6)$$

(2) If m is an odd number, we have

$$L(g^{(n)}(t)) = -s^{n-1}g(0)\Theta(-s^n)L(g(t)) \otimes \sum_{k=1}^{n-1} s^{n-(k+1)}g^{(k)}(0), \tag{3.7}$$

such that

$$\otimes = \begin{cases} \Theta, & \text{if the number of (ii)} - \text{differentiable functions } g^{(i)}, \text{provided} \\ & \text{that } i < k \textit{ is an odd} \text{ number} \\ -, & \text{if the number of (ii)} - \text{differentiable functions } g^{(i)}, \text{provided} \\ & \text{that } i < k \textit{ is an even} \text{ number} \end{cases} \tag{3.8}$$

Proof (1) Let $g^{(i_1)}(t), g^{(i_2)}(t), \dots, g^{(i_m)}(t)$ be (ii)-differentiable fuzzy valued functions for $0 \le i_1 < i_2 < \dots < i_m \le n-1$ and m be an even number such that $2 \le m \le n$, then by theorem 3.1(a), we get

$g^{(n)}(t) = (\underline{g}^{(n)}(t,\alpha), \overline{g}^{(n)}(t,\alpha))$,

where $f_\alpha(t) = \underline{g}(t,\alpha)$ and $g_\alpha(t) = \overline{g}(t,\alpha)$. Therefore, we get:

$$\underline{g^{(n)}}(t,\alpha) = \underline{g}^{(n)}(t,\alpha), \overline{g^{(n)}}(t,\alpha) = \overline{g}^{(n)}(t,\alpha). \tag{3.9}$$

Then from (3. 9), we get

$$L(g^{(n)}(t)) = L(\underline{g^{(n)}}(t,\alpha), \overline{g^{(n)}}(t,\alpha))$$

$$= (l(\underline{g}^{(n)}(t,\alpha)), l(\overline{g}^{(n)}(t,\alpha))). \tag{3. 10}$$

We know from the ordinary differential equations that:

$$l(\underline{g}^{(n)}(t,\alpha)) = s^n l(\underline{g}(t,\alpha)) - \sum_{i=0}^{n-1} s^{n-(i+1)}\underline{g}^{(i)}(0,\alpha). \tag{3.11}$$

Equation (3.11) can be written as:

$$l(\underline{g}^{(n)}(t,\alpha)) = s^n l(\underline{g}(t,\alpha)) - s^{n-1}\underline{g}(0,\alpha) - \sum_{k=1}^{i_1} s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha) - \sum_{k=i_1+1}^{i_2} s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha)$$

$$- \sum_{k=i_2+1}^{i_3} s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha) - \dots - \sum_{k=i_m+1}^{n-1} s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha). \tag{3.12}$$

In a similar manner, we can get

$$l(\overline{g}^{(n)}(t,\alpha)) = s^n l(\overline{g}(t,\alpha)) - s^{n-1}\overline{g}(0,\alpha) - \sum_{k=1}^{i_1} s^{n-(k+1)}\overline{g}^{(k)}(0,\alpha) - \sum_{k=i_1+1}^{i_2} s^{n-(k+1)}\overline{g}^{(k)}(0,\alpha)$$

$$- \sum_{k=i_2+1}^{i_3} s^{n-(k+1)}\overline{g}^{(k)}(0,\alpha) - \dots - \sum_{k=i_m+1}^{n-1} s^{n-(k+1)}\overline{g}^{(k)}(0,\alpha). \tag{3.13}$$

Since $0 \le i_1 < i_2 < \dots < i_m \le n-1$ we can apply theorem 3.1 for each $g^{(k)}(t)$ where $1 \le k \le n-1$ as follows:

$$\begin{aligned}
&\underline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \overline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha),\ 1 \le k \le i_1, \\
&\overline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \underline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha), i_1+1 \le k \le i_2, \\
&\underline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \overline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha),\ i_2+1 \le k \le i_3, \\
&\vdots \\
&\underline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \overline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha),\ i_m+1 \le k \le n-1.
\end{aligned} \tag{3.14}$$

The last one of the equations in (3.14) yields from theorem 3.1(a) because m is an even number. Using (3.12), (3.13) and (3.14), equation (3.10) becomes

$$L(g^{(n)}(t)) = (s^n \ l(\underline{g}(t,\alpha)) - s^{n-1}\ \underline{g}(0,\alpha) - \sum_{k=1}^{i_1} s^{n-(k+1)}\ \underline{g^{(k)}}(0,\alpha) - \sum_{k=i_1+1}^{i_2} s^{n-(k+1)}\ \overline{g^{(k)}}(0,\alpha) -$$

$$\sum_{k=i_2+1}^{i_3} s^{n-(k+1)}\ \underline{g^{(k)}}(0,\alpha) - \cdots - \sum_{k=i_m+1}^{n-1} s^{n-(k+1)}\ \underline{g^{(k)}}(0,\alpha), s^n\ l(\overline{g}(t,\alpha)) - s^{n-1}\overline{g}(0,\alpha)$$

$$-\sum_{k=1}^{i_1} s^{n-(k+1)}\ \overline{g^{(k)}}(0,\alpha) - \sum_{k=i_1+1}^{i_2} s^{n-(k+1)}\ \underline{g^{(k)}}(0,\alpha) - \sum_{k=i_2+1}^{i_3} s^{n-(k+1)}\ \overline{g^{(k)}}(0,\alpha) - \cdots -$$

$$\sum_{k=i_m+1}^{n-1} s^{n-(k+1)}\overline{g^{(k)}}(0,\alpha)).$$

$$= s^n L(g(t)) \ominus s^{n-1} g(0) \ominus \sum_{k=1}^{i_1} s^{n-(k+1)} g^{(k)}(0) - \sum_{k=i_1+1}^{i_2} s^{n-(k+1)} g^{(k)}(0)$$

$$\ominus \sum_{k=i_2+1}^{i_3} s^{n-(k+1)} g^{(k)}(0) - \cdots - \ominus \sum_{k=i_m+1}^{n-1} s^{n-(k+1)} g^{(k)}(0)$$

$$= s^n L(g(t)) \ominus s^{n-1} g(0) \otimes \sum_{k=1}^{n-1} s^{n-(k+1)} g^{(k)}(0),$$

where $\otimes$ is defined as in (3. 6) . Then, the theorem is true for any even number m such that $2 \le m \le n$

We note that if $m = 0$ we have $g(t), g'(t), \ldots, g^{(n-1)}$(t) are (i)-differentiable functions, then equation (3.10) becomes:

$$L(g^{(n)}(t)) = (s^n \ l(\underline{g}(t,\alpha)) - \sum_{k=0}^{n-1} s^{n-(k+1)}\ \underline{g^{(k)}}(0,\alpha),\ s^n\ l(\overline{g}(t,\alpha)) - \sum_{k=0}^{n-1} s^{n-(k+1)}\ \overline{g^{(k)}}(0,\alpha))$$

$$= s^n L(g(t)) \ominus \sum_{k=0}^{n-1} s^{n-(k+1)} g^{(k)}(0).$$

It is clear that the above relation reconciles with the relations (3.5) and (3.6).
Therefore, the theorem is true for any even number m such that $0 \le m \le n$

Proof (2) Let $g^{(i_1)}(t), g^{(i_2)}(t), \ldots, g^{(i_m)}$(t) be (ii)-differentiable fuzzy valued functions for $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ and m be an odd number such that $1 \le m \le n$ and, then by theorem 3.1(b) we get

$g^{(n)}(t) = (\overline{g}^{(n)}(t,\alpha), \underline{g}^{(n)}(t,\alpha))$.

Therefore, we get:

$$\underline{g^{(n)}}(t,\alpha) = \overline{g}^{(n)}(t,\alpha), \overline{g^{(n)}}(t,\alpha) = \underline{g}^{(n)}(t,\alpha). \quad (3.15)$$

Therefore, from (3.15) we get

$$L(g^{(n)}(t)) = L(\underline{g^{(n)}}(t,\alpha), \overline{g^{(n)}}(t,\alpha))$$

$$= (l(\overline{g}^{(n)}(t,\alpha)), l(\underline{g}^{(n)}(t,\alpha))). \quad (3.16)$$

Since $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ we can apply theorem 3.1 for each $g^{(k)}(t)$ where $1 \le k \le n-1$ as follows:

$$\underline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \overline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha)\ ,\ 1 \le k \le i_1,$$

$$\overline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \underline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha)\ ,\ i_1 + 1 \le k \le i_2,$$

$$\underline{g}^{(k)}(0,\alpha) = \underline{g^{(k)}}(0,\alpha),\ \overline{g}^{(k)}(0,\alpha) = \overline{g^{(k)}}(0,\alpha),\ i_2 + 1 \le k \le i_3, \quad (3.17)$$

$\vdots$

$\bar{g}^{(k)}(0,\alpha)=\underline{g}^{(k)}(0,\alpha),\ \underline{g}^{(k)}(0,\alpha)=\overline{g^{(k)}}(0,\alpha),\ \ i_m+1\le k\le n-1.$

The last one of the equations in (3.17) yields from theorem 3.1(b) because m is an odd number. Using (3. 12), (3.13) and (3. 17), equation (3. 16) becomes

$$L(g^{(n)}(t))=s^n\ l(\bar{g}(t,\alpha))-s^{n-1}\ \bar{g}(0,\alpha)-\sum_{k=1}^{i_1}s^{n-(k+1)}\ \overline{g^{(k)}}(0,\alpha)-\sum_{k=i_1+1}^{i_2}s^{n-(k+1)}\ \underline{g}^{(k)}(0,\alpha)-$$
$$\sum_{k=i_2+1}^{i_3}s^{n-(k+1)}\overline{g^{(k)}}(0,\alpha)-\cdots-\sum_{k=i_m+1}^{n-1}s^{n-(k+1)}\ \underline{g}^{(k)}(0,\alpha),s^n l(\underline{g}(t,\alpha))-s^{n-1}\underline{g}(0,\alpha)$$
$$-\sum_{k=1}^{i_1}s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha)-\sum_{k=i_1+1}^{i_2}s^{n-(k+1)}\overline{g^{(k)}}(0,\alpha)-\sum_{k=i_2+1}^{i_3}s^{n-(k+1)}\underline{g}^{(k)}(0,\alpha)-\cdots-$$
$$\sum_{k=i_m+1}^{n-1}s^{n-(k-1)}\overline{g^{(k)}}(0,\alpha)).$$

Thus

$$L(g^{(n)}(t))=-s^{n-1}g(0)\ominus(-s^n)L(g(t))-\sum_{k=1}^{i_1}s^{n-(k+1)}g^{(k)}(0)\ominus\sum_{k=i_1+1}^{i_2}s^{n-(k+1)}g^{(k)}(0)$$
$$-\sum_{k=i_2+1}^{i_3}s^{n-(k+1)}g^{(k)}(0)\ \ominus\cdots\ominus\sum_{k=i_m+1}^{n-1}s^{n-(k+1)}g^{(k)}(0)$$
$$=-s^{n-1}g(0)\ominus(-s^n)L(g(t))\otimes\sum_{k=1}^{n-1}s^{n-(1+k)}g^{(k)}(0),$$

where $\otimes$ is defined as in (3. 8)

Remark 3.4 If we put $n=1,2,3,4$ in theorem 3.3, we get the same results given in [7,8,9,9] respectively.

Remark 3.5 By applying theorem 3.3, we note that: There are 2^n cases for fuzzy Laplace transforms for $g^{(n)}(t), n\in Z^+$. Also, we have

$$\sum_{k=0}^{n}\binom{n}{k}=2^n,$$

where $\binom{n}{k}$ is the number of cases that contain k functions of the type (ii)-differentiable functions among the functions $g(t),g'(t),\ldots,g^{(n-1)}(t)$.

4. Application of the Generalization

In this section, we present an example of the third order to show the validity of the generalization given in theorem 3.3.

Example 4.1 Consider the following third-order FIVP:

$$y'''(t) = -y''(t) - 3y'(t) + 5y(t), \tag{4.1}$$

$$y(0) = (\frac{3}{4}+\frac{r}{4}, \frac{5}{4}-\frac{r}{4}),\ \ y'(0) = (\frac{3}{2}+\frac{r}{2}, \frac{5}{2}-\frac{r}{2}),\ \ y''(0) = (\frac{15}{4}+\frac{r}{4}, \frac{17}{4}-\frac{r}{4}).$$

We note that:

$$\underline{y}(0,r) = \frac{3}{4}+\frac{r}{4}, \overline{y}(0,r) = \frac{5}{4}-\frac{r}{4}, \underline{y}'(0,r) = \frac{3}{2}+\frac{r}{2}, \overline{y}'(0,r) = \frac{5}{2}-\frac{r}{2}, \underline{y}''(0,r) = \frac{15}{4}+\frac{r}{4}, \overline{y}''(0,r) = \frac{17}{4}-\frac{r}{4}$$

By taking fuzzy Laplace transform for both sides of equation (4.1), we get:

$$L\left[y'''(t)\right] = L\left[-y''(t) - 3y'(t) + 5y(t)\right] \tag{4.2}$$

Now, we shall use theorem 3.3 to find Laplace transform for each term in equation (4.2), therefore we have $2^3 = 8$ cases as follows:

Case 1 Let us consider $y(t), y'(t)$ and $y''(t)$ be (i)-differentiable. Then equation (4.2) becomes:

$$s^3 L(y(t)) \ominus s^2 y(0) \ominus sy'(t) \ominus y''(0) = -[s^2 L(y(t)) \ominus sy(0) \ominus y'(t)] - 3\,[sL(y(t)) \ominus y(0)]$$
$$+5L(y(t)),$$

then, we get the system:

$$\begin{aligned} &(s^3-5)l(\underline{y}(t,r)) + (s^2+3s)l(\overline{y}(t,r)) = (\frac{3}{4}+\frac{r}{4})s^2 + (\frac{11}{4}+\frac{r}{4})s + 10 - r, \\ &(s^3-5)l(\overline{y}(t,r)) + (s^2+3s)l(\underline{y}(t,r)) = (\frac{5}{4}-\frac{r}{4})s^2 + (\frac{13}{4}-\frac{r}{4})s + 8 + r. \end{aligned} \tag{4.3}$$

The solution of system (4.3) is as follows:

$$l(\underline{y}(t,r)) = \frac{(r+3)s^5 + (2r+6)s^4 + 12s^3 - (6r+86)s^2 - (17r+151)s + 20r - 200}{4(s^3+s^2+3s-5)(s^3-s^2-3s-5)} \ ,$$

$$l(\overline{y}(t,r)) = \frac{(5-r)s^5 + (10-2r)s^4 + 12s^3 + (6r-98)s^2 + (17r-185)s - 20r - 160}{4(s^3+s^2+3s-5)(s^3-s^2-3s-5)} \ .$$

To make a cubic partition for $l(\underline{y}(t,r))$ and $l(\overline{y}(t,r))$, we suppose that:

$$\frac{a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}{4(s^3+s^2+3s-5)(s^3+b_1 s^2+b_2 s+b_3)} = \frac{1}{4}[\frac{As^2+Bs+C}{s^3+s^2+3s-5} + \frac{Ds^2+Es+F}{s^3+b_1 s^2+b_2 s+b_3}], \tag{4.4}$$

yields the system:

$$\begin{aligned} &A + D = a_5, \\ &b_1 A + B + D + E = a_4, \\ &b_2 A + b_1 B + C + 3D + E + F = a_3, \\ &b_3 A + b_2 B + b_1 C - 5D + 3E + F = a_2, \\ &b_3 B + b_2 C - 5E + 3F = a_1, \\ &b_3 C - 5F = a_0. \end{aligned} \tag{4.5}$$

To get the cubic partition for $l(\underline{y}(t,r))$, we put $a_0 = 20r - 200, a_1 = -151 - 17r$, $a_2 = -86 - 6r$, $a_3 = 12, a_4 = 2r + 6, a_5 = r + 3$, $b_1 = -1$, $b_2 = -3$ and $b_3 = -5$ in the system (4.5), then we get:

$$l(\underline{y}(t,r)) = \frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)} + \frac{(r-1)(s^2 + s - 4)}{4(s^3 - s^2 - 3s - 5)}. \quad (4.6)$$

Similarly, to get the partition for $l(\bar{y}(t,r))$, we put $a_0 = -20r - 160$, $a_1 = 17r - 185$, $a_2 = 6r - 98$, $a_3 = 12$, $a_4 = 10 - 2r$, $a_5 = 5 - r$, $b_1 = -1$, $b_2 = -3$ and $b_3 = -5$ in the system (4.5), then:

$$l(\bar{y}(t,r)) = \frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)} - \frac{(r-1)(s^2 + s - 4)}{4(s^3 - s^2 - 3s - 5)}. \quad (4.7)$$

Now, by finding a root for the cubic polynomial given in (4.6) and (4.7), we get:

$$\begin{aligned} \underline{y}(t,r) &= l^{-1}\left(\frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)}\right) + \frac{r-1}{4} l^{-1}\left(\frac{s^2 + s - 4}{(s-a)(s^2 + bs + c)}\right) \\ \bar{y}(t,r) &= l^{-1}\left(\frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)}\right) - \frac{r-1}{4} l^{-1}\left(\frac{s^2 + s - 4}{(s-a)(s^2 + bs + c)}\right) \end{aligned} . \quad (4.8)$$

where

$$\begin{aligned} a &= \frac{1}{3} + \frac{1}{3}\sqrt[3]{82 + 6\sqrt{159}} + \frac{1}{3}\sqrt[3]{82 - 6\sqrt{159}}, \\ \text{b} &= \frac{-2}{3} + \frac{1}{3}\sqrt[3]{82 + 6\sqrt{159}} + \frac{1}{3}\sqrt[3]{82 - 6\sqrt{159}}, \\ \text{c} &= -1 - \frac{1}{9}\sqrt[3]{82 + 6\sqrt{159}} - \frac{1}{9}\sqrt[3]{82 - 6\sqrt{159}} + \frac{1}{9}(82 + 6\sqrt{159})^{\frac{2}{3}} + \frac{1}{9}(82 - 6\sqrt{159})^{\frac{2}{3}}. \end{aligned} \quad (4.9)$$

To find l^{-1} in the system (4.8), we put:

$$\frac{d_2 s^2 + d_1 s + d_0}{(s-a)(s^2 + bs + c)} = \frac{(-d_0 - ad_1 + cd_2 + abd_2)s + (d_1 c - d_0 a - d_0 b + ad_2 c)}{(a^2 + ab + c)(s^2 + bs + c)} + \frac{d_2 a^2 + d_1 a + d_0}{a^2 + ab + c}\frac{1}{s-a}.$$

Then:

$$l^{-1}\left(\frac{d_2 s^2 + d_1 s + d_0}{(s-a)(s^2 + bs + c)}\right) = \frac{d_2 a^2 + d_1 a + d_0}{a^2 + ab + c} e^{at} - \frac{d_0 + ad_1 - cd_2 - abd_2}{a^2 + ab + c} e^{-\frac{b}{2}t} \cosh\sqrt{\frac{b^2}{4} - c}\,t$$

$$+ \frac{d_0 + ad_1 - cd_2 - abd_2}{(a^2 + ab + c)\sqrt{\frac{b^2}{4} - c}}\left(\frac{b}{2} - \frac{ad_0 + bd_0 - cd_1 - acd_2}{d_0 + ad_1 - cd_2 - abd_2}\right) e^{-\frac{b}{2}t} \sinh\sqrt{\frac{b^2}{4} - c}\,t. \quad (4.10)$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.8) becomes:

$$\underline{y}(t,r) = \frac{13}{8}e^t - \frac{5}{8}e^{-t}\cos 2t - \frac{1}{8}e^{-t}\sin 2t + (r-1)(c_1 e^{at} - c_2 e^{\frac{-b}{2}t}\cosh c_4\,t + c_3 e^{\frac{-b}{2}t}\sinh c_4\,t),$$

$$\bar{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(c_1e^{at}-c_2e^{\frac{-b}{2}t}\cosh c_4\, t+c_3e^{\frac{-b}{2}t}\sinh c_4\, t),$$

where

$$c_1=\frac{-12+5\sqrt[3]{82+6\sqrt{159}}+5\sqrt[3]{82-6\sqrt{159}}+(82+6\sqrt{159})^{\frac{2}{3}}+(82-6\sqrt{159})^{\frac{2}{3}}}{4\,[30+3(82+6\sqrt{159})^{\frac{2}{3}}+3(82-6\sqrt{159})^{\frac{2}{3}}]},$$

$$c_2=\frac{-42+5\sqrt[3]{82+6\sqrt{159}}+5\sqrt[3]{82-6\sqrt{159}}-2(82+6\sqrt{159})^{\frac{2}{3}}-2(82-6\sqrt{159})^{\frac{2}{3}}}{4\,[30+3(82+6\sqrt{159})^{\frac{2}{3}}+3(82-6\sqrt{159})^{\frac{2}{3}}]},$$

$$c_3=\frac{6[22\sqrt[3]{82+6\sqrt{159}}+22\sqrt[3]{82-6\sqrt{159}}+5(82+6\sqrt{159})^{\frac{2}{3}}+5(82-6\sqrt{159})^{\frac{2}{3}}]\,c_4}{4\,[30+3(82+6\sqrt{159})^{\frac{2}{3}}+3(82-6\sqrt{159})^{\frac{2}{3}}][20-(82+6\sqrt{159})^{\frac{2}{3}}-(82-6\sqrt{159})^{\frac{2}{3}}]}$$

$$c_4=\frac{1}{6}\sqrt{60-3(82+6\sqrt{159})^{\frac{2}{3}}-3(82-6\sqrt{159})^{\frac{2}{3}}}.$$

and a,b and c are defined as in (4.5).

Case 2 Let $y'(t)$ and $y''(t)$ be (i)-differentiable and $y(t)$ be (ii)-differentiable. Then equation (4.2) becomes:

$$-s^2y(0)\ominus(-s^3)L(y(t))\ominus sy'(t)\ominus y''(0)=-[-sy(0)\ominus(-s^2)L(y(t))\ominus y'(t)]-3\,[-y(0)\ominus (-s)L(y(t))]+5L(y(t)),$$

then, we get the system:

$$\begin{aligned}&s^3l(\underline{y}(t,r))+(s^2+3s-5)l(\bar{y}(t,r))=(\frac{3}{4}+\frac{r}{4})s^2+(\frac{15}{4}-\frac{3r}{4})s+\frac{19}{2}-\frac{r}{2},\\&s^3l(\bar{y}(t,r))+(s^2+3s-5)l(\underline{y}(t,r))=(\frac{5}{4}-\frac{r}{4})s^2+(\frac{9}{4}+\frac{3r}{4})s+\frac{17}{2}+\frac{r}{2}.\end{aligned}\tag{4.11}$$

By achieving the same steps given in case 1, we get:

$$\begin{aligned}&\underline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})+\frac{r-1}{4}l^{-1}(\frac{s^2-3s-2}{(s-a)(s^2+bs+c)}),\\&\bar{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})-\frac{r-1}{4}l^{-1}(\frac{s^2-3s-2}{(s-a)(s^2+bs+c)}).\end{aligned}\tag{4.12}$$

where

$$a=\frac{1}{3}+\frac{1}{3}\sqrt[3]{-53+3\sqrt{201}}+\frac{1}{3}\sqrt[3]{-53-3\sqrt{201}},$$
$$b=\frac{-2}{3}+\frac{1}{3}\sqrt[3]{-53+3\sqrt{201}}+\frac{1}{3}\sqrt[3]{-53-3\sqrt{201}}, \quad (4.13)$$
$$c=-1-\frac{1}{9}\sqrt[3]{-53+3\sqrt{201}}-\frac{1}{9}\sqrt[3]{-53-3\sqrt{201}}+\frac{1}{9}(-53+3\sqrt{201})^{\frac{2}{3}}+\frac{1}{9}(-53-3\sqrt{201})^{\frac{2}{3}}.$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.12) becomes:

$$\underline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t+(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$
$$\bar{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

where

$$c_1=\frac{-6-7\sqrt[3]{-53+3\sqrt{201}}-7\sqrt[3]{-53-3\sqrt{201}}+(-53+3\sqrt{201})^{\frac{2}{3}}+(-53-3\sqrt{201})^{\frac{2}{3}}}{4[30+3(-53+3\sqrt{201})^{\frac{2}{3}}+3(-53-3\sqrt{201})^{\frac{2}{3}}]},$$

$$c_2=\frac{-36-7\sqrt[3]{-53+3\sqrt{201}}-7\sqrt[3]{-53-3\sqrt{201}}-2(-53+3\sqrt{201})^{\frac{2}{3}}-2(-53-3\sqrt{201})^{\frac{2}{3}}}{4\,[30+3(-53+3\sqrt{201})^{\frac{2}{3}}+3(-53-3\sqrt{201})^{\frac{2}{3}}]},$$

$$c_3=\frac{6[16\sqrt[3]{-53+3\sqrt{201}}+16\sqrt[3]{-53-3\sqrt{201}}-7(-53+3\sqrt{201})^{\frac{2}{3}}-7(-53-3\sqrt{201})^{\frac{2}{3}}]\,c_4}{4\,[30+3(-53+3\sqrt{201})^{\frac{2}{3}}+3(-53+3\sqrt{201})^{\frac{2}{3}}][20-(-53+3\sqrt{201})^{\frac{2}{3}}-(-53-3\sqrt{201})^{\frac{2}{3}}]}$$

$$c_4=\frac{1}{6}\sqrt{60-3(-53+3\sqrt{201})^{\frac{2}{3}}-3(-53-3\sqrt{201})^{\frac{2}{3}}}\ .$$

and a,b and c are defined as in (4.13).

Case 3 Let $y(t)$ and $y''(t)$ be (i)-differentiable and $y'(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$-s^2y(0)\ominus(-s^3)L(y(t))-sy'(t)\ \ \ominus y''(0)=-[-sy(0)\ominus(-s^2)L(y(t))-y'(t)]-3\,[sL(y(t))\ominus y(0)]$$
$$+5L(y(t)),$$

then, we get the system:

$$(s^3+3s)l(\underline{y}(t,r))+(s^2-5)l(\bar{y}(t,r))=(\frac{3}{4}+\frac{r}{4})s^2+(\frac{11}{4}+\frac{r}{4})s+9,$$
$$(s^3+3s)l(\bar{y}(t,r))+(s^2-5)l(\underline{y}(t,r))=(\frac{5}{4}-\frac{r}{4})s^2+(\frac{13}{4}-\frac{r}{4})s+9. \quad (4.14)$$

The solution of system (4.14) is as follows:

$$l(\underline{y}(t,r))=\frac{13}{8(s-1)}-\frac{5s+7}{8(s^2+2s+5)}+\frac{(r-1)s}{4(s^2-2s+5)},$$

$$l(\overline{y}(t,r))=\frac{13}{8(s-1)}-\frac{5s+7}{8(s^2+2s+5)}-\frac{(r-1)s}{4(s^2-2s+5)}.$$

Then, we get $r-$cut representation of solution as follows:

$$\underline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t+(r-1)(\frac{1}{4}e^{t}\cos 2t+\frac{1}{8}e^{t}\sin 2t),$$

$$\overline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(\frac{1}{4}e^{t}\cos 2t+\frac{1}{8}e^{t}\sin 2t).$$

Case 4 Let $y(t)$ and $y'(t)$ be (i)-differentiable and $y''(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$-s^2y(0)\Theta(-s^3)L(y(t))-sy'(t)-y''(0)=-[s^2L(y(t))\Theta sy(0)\Theta y'(t)]-3[sL(y(t))\Theta y(0)]$$
$$+5L(y(t)),$$

then, we get the system:

$$\begin{aligned}&(s^3+s^2+3s)l(\underline{y}(t,r))-5l(\overline{y}(t,r))=(\frac{3}{4}+\frac{r}{4})s^2+(\frac{9}{4}+\frac{3r}{4})s+\frac{15}{2}+\frac{3r}{2},\\&(s^3+s^2+3s)l(\overline{y}(t,r))-5l(\underline{y}(t,r))=(\frac{5}{4}-\frac{r}{4})s^2+(\frac{15}{4}-\frac{3r}{4})s+\frac{21}{2}-\frac{3r}{2}.\end{aligned} \quad (4.15)$$

By achieving the same steps given in case 1, we get:

$$\begin{aligned}&\underline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})+\frac{r-1}{4}l^{-1}(\frac{s^2+3s+6}{(s-a)(s^2+bs+c)}),\\&\overline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})-\frac{r-1}{4}l^{-1}(\frac{s^2+3s+6}{(s-a)(s^2+bs+c)}).\end{aligned} \quad (4.16)$$

where

$$\begin{aligned}&a=\frac{-1}{3}+\frac{1}{3}\sqrt[3]{-55+3\sqrt{393}}+\frac{1}{3}\sqrt[3]{-55-3\sqrt{393}},\\&\text{b}=\frac{2}{3}+\frac{1}{3}\sqrt[3]{-55+3\sqrt{393}}+\frac{1}{3}\sqrt[3]{-55-3\sqrt{393}},\\&\text{c}=1+\frac{1}{9}\sqrt[3]{-55+3\sqrt{393}}+\frac{1}{9}\sqrt[3]{-55-3\sqrt{393}}+\frac{1}{9}(-55+3\sqrt{393})^{\frac{2}{3}}+\frac{1}{9}(-55-3\sqrt{393})^{\frac{2}{3}}.\end{aligned} \quad (4.17)$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.16) becomes:

$$\underline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t+(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

$$\overline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

where

$$c_1 = \frac{30+7\sqrt[3]{-55+3\sqrt{393}}+7\sqrt[3]{-55-3\sqrt{393}}+(-55+3\sqrt{393})^{\frac{2}{3}}+(-55-3\sqrt{393})^{\frac{2}{3}}}{4[-24+3(-55+3\sqrt{393})^{\frac{2}{3}}+3(-55-3\sqrt{393})^{\frac{2}{3}}]},$$

$$c_2 = \frac{54+7\sqrt[3]{-55+3\sqrt{393}}+7\sqrt[3]{-55-3\sqrt{393}}-2(-55+3\sqrt{393})^{\frac{2}{3}}-2(-55-3\sqrt{393})^{\frac{2}{3}}}{4[-24+3(-55+3\sqrt{393})^{\frac{2}{3}}+3(-55-3\sqrt{393})^{\frac{2}{3}}]},$$

$$c_3 = \frac{6[-38\sqrt[3]{-55+3\sqrt{393}}-38\sqrt[3]{-55-3\sqrt{393}}+7(-55+3\sqrt{393})^{\frac{2}{3}}+7(-55-3\sqrt{393})^{\frac{2}{3}}]\, c_4}{4[-24+3(-55+3\sqrt{393})^{\frac{2}{3}}+3(-55-3\sqrt{393})^{\frac{2}{3}}][-16-(-55+3\sqrt{393})^{\frac{2}{3}}-(-55-3\sqrt{393})^{\frac{2}{3}}]}$$

$$c_4 = \frac{1}{6}\sqrt{-48-3(-55+3\sqrt{393})^{\frac{2}{3}}-3(-55-3\sqrt{393})^{\frac{2}{3}}}.$$

and a,b and c are defined as in (4.17).

Case 5 Let $y''(t)$ be (i)-differentiable and $y(t)$ and $y'(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$s^3L(y(t))\ominus s^2y(0)-sy'(t)\ominus y''(0)=-[s^2L(y(t))\ominus sy(0)-y'(t)]-3[-y(0)\ominus(-s)L(y(t))]$$
$$+5L(y(t)),$$

then, we get the system:

$$\begin{aligned}
&(s^3+3s-5)l(\underline{y}(t,r))+s^2\ l(\overline{y}(t,r))=(\frac{3}{4}+\frac{r}{4})s^2+(\frac{15}{4}-\frac{3r}{4})s+\frac{15}{2}+\frac{3r}{2},\\
&(s^3+3s-5)l(\overline{y}(t,r))+s^2\ l(\underline{y}(t,r))=(\frac{5}{4}-\frac{r}{4})s^2+(\frac{9}{4}+\frac{3r}{4})s+\frac{21}{2}-\frac{3r}{2}.
\end{aligned} \tag{4.18}$$

By achieving the same steps given in case 1, we get:

$$\begin{aligned}
&\underline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})+\frac{r-1}{4}l^{-1}(\frac{s^2-3s+6}{(s-a)(s^2+bs+c)}),\\
&\overline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})-\frac{r-1}{4}l^{-1}(\frac{s^2-3s+6}{(s-a)(s^2+bs+c)}).
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
&a=\frac{1}{3}+\frac{1}{3}\sqrt[3]{55+3\sqrt{393}}+\frac{1}{3}\sqrt[3]{55-3\sqrt{393}},\\
&\text{b}=\frac{-2}{3}+\frac{1}{3}\sqrt[3]{55+3\sqrt{393}}+\frac{1}{3}\sqrt[3]{55-3\sqrt{393}},\\
&\text{c}=1-\frac{1}{9}\sqrt[3]{55+3\sqrt{393}}-\frac{1}{9}\sqrt[3]{55-3\sqrt{393}}+\frac{1}{9}(55+3\sqrt{393})^{\frac{2}{3}}+\frac{1}{9}(55-3\sqrt{393})^{\frac{2}{3}}.
\end{aligned} \tag{4.20}$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.19) becomes:

$$\underline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t+(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

$$\overline{y}(t,r)=\frac{13}{8}e^{t}-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

where

$$c_1=\frac{30-7\sqrt[3]{55+3\sqrt{393}}-7\sqrt[3]{55-3\sqrt{393}}+(55+3\sqrt{393})^{\frac{2}{3}}+(55-3\sqrt{393})^{\frac{2}{3}}}{4[-24+3(55+3\sqrt{393})^{\frac{2}{3}}+3(55-3\sqrt{393})^{\frac{2}{3}}]},$$

$$c_2=\frac{54-7\sqrt[3]{55+3\sqrt{393}}-7\sqrt[3]{55-3\sqrt{393}}-2(55+3\sqrt{393})^{\frac{2}{3}}-2(55-3\sqrt{393})^{\frac{2}{3}}}{4[-24+3(55+3\sqrt{393})^{\frac{2}{3}}+3(55-3\sqrt{393})^{\frac{2}{3}}]},$$

$$c_3=\frac{6[-38\sqrt[3]{55+3\sqrt{393}}-38\sqrt[3]{55-3\sqrt{393}}-7(55+3\sqrt{393})^{\frac{2}{3}}-7(55-3\sqrt{393})^{\frac{2}{3}}]\,c_4}{4\,[-24+3(55+3\sqrt{393})^{\frac{2}{3}}+3(55-3\sqrt{393})^{\frac{2}{3}}][-12-(55+3\sqrt{393})^{\frac{2}{3}}-(55-3\sqrt{393})^{\frac{2}{3}}]}$$

$$c_4=\frac{1}{6}\sqrt{-36-3(55+3\sqrt{393})^{\frac{2}{3}}-3(55-3\sqrt{393})^{\frac{2}{3}}}.$$

and a,b and c are defined as in (4.20).

Case 6 Let $y'(t)$ be (i)-differentiable and $y(t)$ and $y''(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$s^3L(y(t))\ominus s^2y(0)-sy'(t)-y''(0)=-[-sy(0)\ominus(-s^2)L(y(t))\ominus y'(t)]-3\,[-y(0)\ominus(-s)L(y(t))]$$
$$+5L(y(t)),$$

then, we get:

$$l(\underline{y}(t,r))=\frac{13}{8(s-1)}+\frac{(2r-7)s-7}{8(s^2+2s+5)},$$

$$l(\overline{y}(t,r))=\frac{13}{8(s-1)}-\frac{(2r+3)s+7}{8(s^2+2s+5)}.$$

Then, we get $r-$cut representation of solution as follows:

$$\underline{y}(t,r)=\frac{13}{8}e^{t}-\frac{7}{8}e^{-t}\cos 2t+r(\frac{1}{4}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t),$$

$$\overline{y}(t,r)=\frac{13}{8}e^{t}-\frac{3}{8}e^{-t}\cos 2t-\frac{1}{4}e^{-t}\sin 2t-r(\frac{1}{4}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t).$$

Case 7 Let $y(t)$ be (i)-differentiable and $y'(t)$ and $y''(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$s^3L(y(t))\ominus s^2y(0)\ominus sy'(t)-y''(0)=-[-sy(0)\ominus(-s^2)L(y(t))-y'(t)]-3\,[sL(y(t))\ominus y(0)]$$
$$+5L(y(t)),$$

then, we get the system:

$$(s^3+s^2-5)l(\underline{y}(t,r))+3s\ l(\overline{y}(t,r))=(\frac{3}{4}+\frac{r}{4})s^2+(\frac{9}{4}+\frac{3r}{4})s+\frac{19}{2}-\frac{r}{2},$$
$$(s^3+s^2-5)l(\overline{y}(t,r))+3s\ l(\underline{y}(t,r))=(\frac{5}{4}-\frac{r}{4})s^2+(\frac{15}{4}-\frac{3r}{4})s+\frac{17}{2}+\frac{r}{2}.$$
(4.21)

By achieving the same steps given in case 1, we get:

$$\underline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})+\frac{r-1}{4}l^{-1}(\frac{s^2+3s-2}{(s-a)(s^2+bs+c)}),$$
$$\overline{y}(t,r)=l^{-1}(\frac{s^2+3s+9}{(s-1)(s^2+2s+5)})-\frac{r-1}{4}l^{-1}(\frac{s^2+3s-2}{(s-a)(s^2+bs+c)}). \quad (4.22)$$

where

$$a=\frac{-1}{3}+\frac{1}{3}\sqrt[3]{53+3\sqrt{201}}+\frac{1}{3}\sqrt[3]{53-3\sqrt{201}},$$
$$b=\frac{2}{3}+\frac{1}{3}\sqrt[3]{53+3\sqrt{201}}+\frac{1}{3}\sqrt[3]{53-3\sqrt{201}}, \quad (4.23)$$
$$c=-1+\frac{1}{9}\sqrt[3]{53+3\sqrt{201}}+\frac{1}{9}\sqrt[3]{53-3\sqrt{201}}+\frac{1}{9}(53+3\sqrt{201})^{\frac{2}{3}}+\frac{1}{9}(53-3\sqrt{201})^{\frac{2}{3}}.$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.22) becomes:

$$\underline{y}(t,r)=\frac{13}{8}e^t-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t+(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$
$$\overline{y}(t,r)=\frac{13}{8}e^t-\frac{5}{8}e^{-t}\cos 2t-\frac{1}{8}e^{-t}\sin 2t-(r-1)(c_1e^{a\,t}-c_2e^{\frac{-b}{2}t}\cosh c_4\,t+c_3e^{\frac{-b}{2}t}\sinh c_4\,t),$$

where

$$c_1=\frac{-6+7\sqrt[3]{53+3\sqrt{201}}+7\sqrt[3]{53-3\sqrt{201}}+(53+3\sqrt{201})^{\frac{2}{3}}+(53-3\sqrt{201})^{\frac{2}{3}}}{4[30+3(53+3\sqrt{201})^{\frac{2}{3}}+3(53-3\sqrt{201})^{\frac{2}{3}}]},$$

$$c_2=\frac{-36+7\sqrt[3]{53+3\sqrt{201}}+7\sqrt[3]{53-3\sqrt{201}}-2(53+3\sqrt{201})^{\frac{2}{3}}-2(53-3\sqrt{201})^{\frac{2}{3}}}{4[30+3(53+3\sqrt{201})^{\frac{2}{3}}+3(53+3\sqrt{201})^{\frac{2}{3}}]},$$

$$c_3=\frac{6[16\sqrt[3]{53+3\sqrt{201}}+16\sqrt[3]{53-3\sqrt{201}}+7(53+3\sqrt{201})^{\frac{2}{3}}+7(53-3\sqrt{201})^{\frac{2}{3}}]\,c_4}{4\,[30+3(53+3\sqrt{201})^{\frac{2}{3}}+3(53-3\sqrt{201})^{\frac{2}{3}}][20-(53+3\sqrt{201})^{\frac{2}{3}}-(53-3\sqrt{201})^{\frac{2}{3}}]}$$

$$c_4=\frac{1}{6}\sqrt{20-3(53+3\sqrt{201})^{\frac{2}{3}}-3(53-3\sqrt{201})^{\frac{2}{3}}}.$$

and a, b and c are defined as in (4.23).

Case 8 Let $y(t)$, $y'(t)$ and $y''(t)$ be (ii)- differentiable. Then equation (4.2) becomes:

$$-s^2 y(0) \Theta s^3 L(y(t)) \Theta sy'(t) - y''(0) = -[s^2 L(y(t)) \Theta sy(0) - y'(t)] - 3[-y(0) \Theta (-s) L(y(t))] + 5L(y(t)),$$

then, we get the system:

$$\begin{aligned} &(s^3 + s^2) l(\underline{y}(t,r)) + (3s - 5) l(\overline{y}(t,r)) = (\frac{3}{4} + \frac{r}{4}) s^2 + (\frac{13}{4} - \frac{r}{4}) s + 10 - r, \\ &(s^3 + s^2) l(\overline{y}(t,r)) + (3s - 5) l(\underline{y}(t,r)) = (\frac{5}{4} - \frac{r}{4}) s^2 + (\frac{11}{4} + \frac{r}{4}) s + 8 + r. \end{aligned} \tag{4.24}$$

By achieving the same steps given in case 1, we get:

$$\begin{aligned} &\underline{y}(t,r) = l^{-1}(\frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)}) + \frac{r-1}{4} l^{-1}(\frac{s^2 - s - 4}{(s-a)(s^2 + bs + c)}), \\ &\overline{y}(t,r) = l^{-1}(\frac{s^2 + 3s + 9}{(s-1)(s^2 + 2s + 5)}) - \frac{r-1}{4} l^{-1}(\frac{s^2 - s - 4}{(s-a)(s^2 + bs + c)}). \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} &a = \frac{-1}{3} + \frac{1}{3}\sqrt[3]{-82 + 12\sqrt{159}} + \frac{1}{3}\sqrt[3]{-82 - 12\sqrt{159}}, \\ &\text{b} = \frac{2}{3} + \frac{1}{3}\sqrt[3]{-82 + 12\sqrt{159}} + \frac{1}{3}\sqrt[3]{-82 - 12\sqrt{159}}, \\ &\text{c} = -1 + \frac{1}{9}\sqrt[3]{-82 + 12\sqrt{159}} + \frac{1}{9}\sqrt[3]{-82 - 12\sqrt{159}} + \frac{1}{9}(-82 + 12\sqrt{159})^{\frac{2}{3}} + \frac{1}{9}(-82 - 12\sqrt{159})^{\frac{2}{3}}. \end{aligned} \tag{4.26}$$

By using relation (4.10), then $r-$cut representation of solution given in equation (4.25) becomes:

$$\underline{y}(t,r) = \frac{13}{8} e^t - \frac{5}{8} e^{-t} \cos 2t - \frac{1}{8} e^{-t} \sin 2t + (r-1)(c_1 e^{at} - c_2 e^{\frac{-b}{2}t} \cosh c_4\, t + c_3 e^{\frac{-b}{2}t} \sinh c_4\, t),$$

$$\overline{y}(t,r) = \frac{13}{8} e^t - \frac{5}{8} e^{-t} \cos 2t - \frac{1}{8} e^{-t} \sin 2t - (r-1)(c_1 e^{at} - c_2 e^{\frac{-b}{2}t} \cosh c_4\, t + c_3 e^{\frac{-b}{2}t} \sinh c_4\, t),$$

where

$$c_1 = \frac{-12 - 5\sqrt[3]{-82 + 12\sqrt{159}} - 5\sqrt[3]{-82 - 12\sqrt{159}} + (-82 + 12\sqrt{159})^{\frac{2}{3}} + (-82 - 12\sqrt{159})^{\frac{2}{3}}}{4[30 + 3(-82 + 12\sqrt{159})^{\frac{2}{3}} + 3(-82 - 12\sqrt{159})^{\frac{2}{3}}]},$$

$$c_2 = \frac{-42 - 5\sqrt[3]{-82+12\sqrt{159}} - 5\sqrt[3]{-82-12\sqrt{159}} - 2(-82+12\sqrt{159})^{\frac{2}{3}} - 2(-82-12\sqrt{159})^{\frac{2}{3}}}{4[30+3(-82+12\sqrt{159})^{\frac{2}{3}} + 3(-82-12\sqrt{159})^{\frac{2}{3}}]},$$

$$c_3 = \frac{6[22\sqrt[3]{-82+12\sqrt{159}} + 22\sqrt[3]{-82-12\sqrt{159}} - 5(-82+12\sqrt{159})^{\frac{2}{3}} - 5(-82-12\sqrt{159})^{\frac{2}{3}}]\, c_4}{4\,[30+3(-82+12\sqrt{159})^{\frac{2}{3}} + 3(-82-12\sqrt{159})^{\frac{2}{3}}][20-(-82+12\sqrt{159})^{\frac{2}{3}} - (-82-12\sqrt{159})^{\frac{2}{3}}]}$$

$$c_4 = \frac{1}{6}\sqrt{60 - 3(-82+12\sqrt{159})^{\frac{2}{3}} - 3(-82-12\sqrt{159})^{\frac{2}{3}}}.$$

and a,b and c are defined as in (4.26).

5.Conclusions

The formula of fuzzy derivatives of any order n, $n \in Z^+$ and the formula of fuzzy Laplace transforms of fuzzy derivatives of any order n, $n \in Z^+$ are found under generalized H-differentiability. To show the validity of the two above generalizations, solutions to FIVP of the third order are provided.

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