On fuzzy i-open sets and fuzzy iα-open sets

المجموعة الضبابية المفتوحة من النوع-i والمجموعة الضبابية المفتوحة من النوع-iα

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Abstract
In this paper, we introduce a new class of fuzzy open sets in fuzzy topological spaces; fuzzy i-open set and fuzzy iα-open set and some of their properties are obtained.
KeywordsFuzzy i-open set, fuzzy iα-open sets, fuzzy i-continuity, fuzzy iα-continuity, fuzzy i-irresolute, fuzzy iα-irresolute, fuzzy i-contra-continuous, fuzzy iα-contra-continuous.

الملخص
في هذا البحث، قدمنا انواعا جديدة من المجاميع الضبابية المفتوحة في الفضاء التبولوجي الضبابي وهي المجموعات الضبابية المفتوحة من النوع-i والمجموعات الضبابية المفتوحة من النوع- iα . وبعض الخصائص التي تم الحصول عليها.

1. Introduction

The fundamental concept of a fuzzy set was introduced in Zadeh [1]. Subsequently, Chang [2] defined the notion of fuzzy topology. An alternative definition of fuzzy topology was given by Lowen [3]. Bin Shahana [10], Singal [9], Azad [6], Singal [9], Ma Bao [7], Parimala ana Devi [12] and Erdal [8] introduced fuzzy semi-open set, fuzzy α-open set, fuzzy semi-continuous, fuzzy α-continuous, fuzzy-irresolute function, fuzzy α-irresolute function, fuzzy semi-contra-continuous, fuzzy α-contra-continuous. In this paper, we define fuzzy i-open set and fuzzy iα-open set via fuzzy topology. Moreover, we define fuzzy i-continuous, fuzzy iα-continuous, fuzzy i-irresolute, fuzzy iα-irresolute, fuzzy i-contra-continuous, fuzzy iα-contra-continuous.

2. Preliminaries

Definition.2.1.[1] Let X be a non-empty set a fuzzy set A in X is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A, for eachx $\in$ X. It is clear that A is completely determined by the set of topples
$A = \{ (x, \mu_A(x)): x \in X\}$.

Definition.2.2.[1] Let $A = \{ (x, \mu_A(x)): x \in X\}$ and $B = \{ (x, \mu_B(x)): x \in X\}$ be two fuzzy sets in X. Then their union $A \vee B$, intersection $A \wedge B$ and complement A^c are also fuzzy sets with the membership functions defined as follows:
i) $\mu_{(A \vee B)}(x) = \max \{ \mu_A(x), \mu_B(x)\}, \forall \ x \in X$,
ii) $\mu_{(A \wedge B)}(x) = \min \{ \mu_A(x), \mu_B(x)\}, \forall \ x \in X$,
iii) $\mu_A{}^c(x) = 1- \mu_A(x), \ \forall \ x \in X$.

Definition.2.3.[1] The symbol I will denote the unit interval [0,1]. Let X be a non-empty set. Now, for the sake of simplicity of notation we will not differentiate between A and μ_A. That is a fuzzy set A in X is a function with domain X and values in I, i.e. an element of I^X. The basic fuzzy sets are the empty set, the whole set the class of all fuzzy sets of X which will be denoted by 0_X , 1_X and I^X, respectively.

Definition.2.4.[2] A family $\tau \subseteq I^X$ of fuzzy sets is called a fuzzy topology for X if it satisfies the following three axioms:

i) 0_X , $1_X \in \tau$,
ii) $\forall$ A, B $\in \tau \Rightarrow$ A $\wedge$ B $\in \tau$,
iii) $\forall (Aj)_{j\in J} \Rightarrow \vee_{j\in J} A_j \in \tau$.

The pair (X, τ) is called a fuzzy topological space or fts, for short. The elements of τ are called fuzzy open sets. The fuzzy closure, the fuzzy interior and the fuzzy complement of any set in A in (X, τ) are denoted by 1_X –A, Int(A) and Cl(A).

Definition.2.5.[11] A fuzzy set which is a fuzzy point with support x∈X and the value λ∈ (0, 1] will be denoted by x_λ . The value of a fuzzy set A for some x∈X will be denoted by A(x). Also, for a fuzzy point x_λ and a fuzzy set A we shall write $x_\lambda \in A$ to mean that $\lambda \leq A(x)$.

Definition.2.6.[4] Let (X, τ) fuzzy topological space and A, B two fuzzy sets then A ≤ B if and only if A(x) ≤ B(x) for all x ∈ X, and A is said to be quasi-coincident with a fuzzy set B, denoted by AqB, if there exists x∈X such that A(x) + B(x) >1.

Definition.2.7.[4] A fuzzy set V in (X, τ) is called a q-neighborhood (q-nbd, for short) of a fuzzy point x_λ if and only if there exists a fuzzy open set U such that x_λ qU ≤ V. We will denote the set of all q-nbd of x_λ in (X, τ) by Nq(x_λ).

Definition.2.8.[10] A fuzzy subset A of a fuzzy topological space (X, τ) is said to be
i) fuzzy semi-open set, if A ≤ Cl(Int(A)) [10],
ii) fuzzy α-open set, if A ≤ Int(Cl(Int(A))) [9].
The family of all fuzzy semi-open (resp. fuzzy α-open) sets of an fuzzy topological space is denoted by FSO(X)(resp. FαO(X)). The complement of fuzzy semi-open (resp. fuzzy α-open) sets of a fuzzy topological space (X, τ) is called fuzzy semi-closed (resp. fuzzy α-closed) sets.

Definition.2.9. Let X and Y be a fuzzy topological spaces, a function f: X→Y is said to be

i) fuzzy semi-continuous if the inverse image of every fuzzy open subset of Y is a fuzzy semi-open subset in X [6],

ii) fuzzy α-continuous if the inverse image of every fuzzy open subset of Y is a fuzzy α-open subset in X [9],
iii) fuzzy-irresolute if the inverse image of every fuzzy semi-open subset of Y is a fuzzy semi-open subset in X [7],

iv) fuzzy α-irresolute if the inverse image of every fuzzy α-open subset of Y is a fuzzy α-open subset
in X [9],
v) fuzzy-contra-continuous if the inverse image of every fuzzy open subset of Y is a fuzzy closed subset in X [8],

vi) fuzzy semi-contra-continuous if the inverse image of every fuzzy open subset of Y is a fuzzy semi-closed subset in X [8],
vii) fuzzy α-contra-continuous if the inverse image of every fuzzy open subset of Y is a fuzzy α-closed subset in X [12].

3. Fuzzy i-open sets and fuzzy iα-open sets

Definition.3.1. A fuzzy subset A of a fuzzy topological space (X, τ) is said to be fuzzy i-open set if there exists a non-empty fuzzy open subset U of X such that $A \leq Cl(A \wedge U)$. The complement of the fuzzy i-open set is called fuzzy i-closed. We denote the family of all fuzzy i-open sets of a fuzzy topological space (X, τ) by FiO(X).

Example.3.1. Let X={a, b} and A, B be a fuzzy sets of X defined as follows:

A(a)=0.1 A(b)=0.9
B(a)=0.5 B(b)=0.8

We put $\tau = \{0_X, 1_X, A\}$. Then B is a fuzzy i-open set.

Theorem.3.1. In a fuzzy topological space (X, τ) the following statements hold:
i) Every fuzzy open set is fuzzy i-open set,
ii) Every fuzzy semi-open set is fuzzy i-open set,
iii) Every fuzzy α-open set is fuzzy i-open set.

Proof. i) It is easy and therefore omitted.

ii) Let A be a fuzzy semi-open set, then we have $A \leq Cl(Int(A))=Cl(Int(A) \wedge A)$, since Int(A) a fuzzy open set and U any fuzzy open set, we choose Int(A)=U, then $A \leq Cl(A \wedge U)$ ($U \in \tau$). Therefore, A is fuzzy i-open set.

iii) Let A be a fuzzy α-open set, then we have $A \leq Int(Cl(Int(A))) \leq Cl(Int(A)) \leq Cl(A \wedge U)$, wher U=Int(A). Therefore, A is fuzzy i-open set.

Remark.3.1. The converse of Theorem 3.1. is not true as show by the following examples.

Example.3.2. Let X={a, b, c} and A, B be a fuzzy sets of X defined as follows:

A(a)=0.2 A(b)=0.7 A(c)=0.4
B(a)=0.7 B(b)=0.9 B(c)=0.1

We put $\tau = \{0_X, 1_X, A\}$. Then B is a fuzzy i-open set, but B is not fuzzy open set.

Example.3.3. Let X={a, b, c} and A, B be a fuzzy sets of X defined as follows:

A(a)=0.2 A(b)=0.7 A(c)=0.4
B(a)=0.7 B(b)=0.9 B(c)=0.5

We put $\tau = \{0_X, 1_X, A\}$. Then B is a fuzzy i-open set, but B is not fuzzy semi-open set and fuzzy α-open set.

Remark.3.2. The intersection of a fuzzy i-open set is not necessary to be a fuzzy i-open set.

Example.3.4. Let X={a, b, c} and A, B, C be a fuzzy sets of X defined as follows:

A(a)=0.6 A(b)=0.3 A(c)=0.4
B(a)=0.5 B(b)=0.9 B(c)=0.6
C(a)=0.5 C(b)=0.1 C(c)=0.7

We put $\tau = \{0_X, 1_X, A, C\}$. Then A and B is a fuzzy i-open set, but $A \wedge B$ is not fuzzy i-open set. Because;
if take $C \in \tau$ then, $A \leq Cl(A \wedge C)$, $B \leq Cl(B \wedge C)$, but $A \wedge B \geq Cl((A \wedge B) \wedge C)$.

Remark.3.3. The union of a fuzzy i-open set is not necessary to be a fuzzy i-open set.

Example.3.5. Let X={a, b, c} and A, B, C be a fuzzy sets of X defined as follows:

A(a)=0.6 A(b)=0.3 A(c)=0.7
B(a)=0.5 B(b)=0.9 B(c)=0.6
C(a)=0.5 C(b)=0.9 C(c)=0.7

We put $\tau = \{0_X, 1_X, A, C, A \wedge C, A \vee C\}$. Then A and B is a fuzzy i-open set, but $A \vee B$ is not fuzzy i-open set. Because; if take $C \in \tau$ then, $A \leq Cl(A \wedge C)$, $B \leq Cl(B \wedge C)$, but $A \vee B \geq Cl((A \vee B) \wedge C)$.

Definition.3.2. A fuzzy subset A of a fuzzy topological space (X, τ) is said to be fuzzy iα-open set if there exists a non-empty subset U of X, U is a fuzzy α-open set, such that $A \leq Cl(A \wedge U)$. The complement of the fuzzy iα-open set is called fuzzy iα-closed. We denote the family of all fuzzy iα-open sets of a fuzzy topological space (X, τ) by FiαO(X).

Example.3.6. in the Example.3.2. B is fuzzy iα-open set.
Theorem.3.2. Every fuzzy i-open set in any fuzzy topological space (X, τ) is a fuzzy iα-open set.
Proof. Let (X, τ) be any fuzzy topological space and $A \le X$ be any fuzzy i-open set. Therefore, $A \le Cl(A \wedge U)$, where $\exists$ $U \in \tau$. Since, every fuzzy open is a fuzzy α-open, then $\exists$ U, fuzzy α-open set. We obtain $A \le Cl(A \wedge U)$, where $\exists$ U, fuzzy α-open set. Thus, A is a fuzzy iα-open set.
Remark.3.4. The following example shows that fuzzy iα-open set need not be fuzzy i-open set.
Example.3.7. Let X={a, b} and A, B, C be a fuzzy sets of X defined as follows:

A(a)=0.2 A(b)=0.8
B(a)=0.5 B(b)=0.6

We put $\tau=\{0_X, 1_X\}$. Then A is a fuzzy iα-open set, but not fuzzy i-open set.
Remark.3.5. The inter section of fuzzy iα-open set is not necessary to be a fuzzy iα-open set as shown in the example.3.4.
Remark.3.6. The union of fuzzy iα-open set is not necessary to be a fuzzy iα-open set as shown in the example.3.5.

4. On decomposition of fuzzy i-continuity and fuzzy iα-continuity

Definition.4.1. Let X and Y be fuzzy topological spaces, a function f: X→Y is said to be fuzzy i-continuous (resp. fuzzy iα-continuous) if the inverse image of every fuzzy open subset of Y is a fuzzy i-open (resp. fuzzy iα-open) subset in X.
Theorem.4.1. Let X and Y be fuzzy topological spaces and function f: X→Y the following statement hold:
i) Every fuzzy -continuous is a fuzzy i-continuous,
ii) Every fuzzy semi-continuous is a fuzzy i-continuous,
iii) Every fuzzy α-continuous is a fuzzy i-continuous.
Proof. This follows from Theorem.3.1. and Definition.4.1.
Theorem.4.2. Every fuzzy i-continuous is a fuzzy iα-continuous.
Proof. The proof is obvious from Theorem.3.2. and Definition.4.1.
Remark.4.1. The converses of Theorem.4.1. and Theorem.4.1. need not true as shown in the follow in examples.
Example.4.1. Let X={a, b, c}, Y={0.1, 0.3, 0.7} and A, B be fuzzy subset defined as follows:

A(a)=0.2 A(b)=0.7 A(c)=0.4
B(0.1)=0.6 B(0.3)=0.3 B(0.7)=0.8

Let $\tau=\{0_X, 1_X, A\}$, $\phi=\{0_Y, 1_Y, B\}$. Then the function f: X→Y defined by

f(a) = 0.1, f(b) = 0.7, f(c) = 0.3

is a fuzzy i-continuous but not fuzzy -continuous, fuzzy semi-continuous and fuzzy α-continuous.
Example.4.2. Let X={a, b, c}, Y={0.3, 0.1, 0.9} and A, B be fuzzy subset defined as follows:

A(a)=0.3 A(b)=0.1 A(c)=0.9
B(0.3)=0.7 B(0.1)=0.3 B(0.9)=0.5

Let $\tau=\{0_X, 1_X\}$, $\phi=\{0_Y, 1_Y, B\}$. Then the function f: X→Y defined by

f(a) = 0.7, f(b) = 0.5, f(c) = 0.3

is a fuzzy iα-continuous but not fuzzy i-continuous.
Definition.4.2. Let X and Y be fuzzy topological spaces, a function f: X→Y is said to be fuzzy i-irresolute (resp. fuzzy iα- irresolute) if the inverse image of every fuzzy i-open(resp. fuzzy iα-open) subset of Y is a fuzzy i-open (resp. fuzzy iα-open) subset in X.
Theorem.4.3. Let X and Y be fuzzy topological spaces and function f: X→Y the following statement hold:
i) Every fuzzy -irresolute is a fuzzy i-irresolute,
ii) Every fuzzy α-irresolute is a fuzzy i-irresolute.
Proof. The proof is obvious from, Theorem.3.1. and Definition.4.2.
Theorem.4.4. Every fuzzy i-irresolute is a fuzzy iα-irresolute.

Proof. The following from Theorem.3.2. and Definition.4.2.

Remark.4.2. The converses of Theorem.4.3. and Theorem.4.4. need not true as shown in the following in examples.

Example.4.3. In Example.4.1. f is a fuzzy i-irresolute but not fuzzy-irresolute.

Example.4.4. In Example.4.2. f is a fuzzy i-irresolute but not fuzzy α-irresolute.

Theorem.4.5. Every fuzzy i-irresolute is a fuzzy i-continuous.

Proof. The following from Theorem.3.1., Definition.4.1. and Definition.4.2.

Theorem.4.6. Every fuzzy iα-irresolute is a fuzzy iα-continuous.

Proof. The following from Theorem.3.2., Definition.4.1. and Definition.4.2.

Remark.4.3. The converses of Theorem.4.5. and Theorem.4.6. need not true as shown in the following in examples.

Example.4.5. Let X={a, b, c}, Y={0.1, 0.5, 0.7} and A, B be fuzzy subset defined as follows:

A(a)=0.8 A(b)=0.2 A(c)=0.4

B(0.1)=0.9 B(0.5)=0.4 B(0.7)=0.7

Let $\tau=\{0_X, 1_X, A\}$, $\phi=\{0_Y, 1_Y\}$. Then the function f: X→Y defined by

f(a) = 0.1, f(b) = 0.5, f(c) = 0.7

is a fuzzy i-continuous and fuzzy iα-continuous but not fuzzy i- irresolute and fuzzy iα- irresolute.

Definition.4.3. Let X and Y be fuzzy topological spaces, a function f: X→Y is said to be fuzzy i-contra-continuous (resp. fuzzy iα-contra-continuous) if the inverse image of every fuzzy open subset of Y is a fuzzy i-closed (resp. fuzzy iα-closed) in X.

Theorem.4.7. Let X and Y be fuzzy topological spaces and function f: X→Y the following statement hold:

i) Every fuzzy -contra-continuous is a fuzzy i-contra-continuous,

ii) Every fuzzy semi-contra-continuous is a fuzzy i-contra-continuous,

iii) Every fuzzy α-contra-continuous is a fuzzy i-contra-continuous.

proof. i) Let f: X→Y be a fuzzy-contra-continuous and V any fuzzy open set in Y. Since f is fuzzy-contra-continuous, then $f^{-1}(V)$ is fuzzy closed sets in X. Since, every fuzzy closed set is a fuzzy i-closed set, then $f^{-1}(V)$ is a fuzzy i-closed set in X. Therefore, f is a fuzzy i-contra-continuous.

ii) Since every fuzzy semi-open set is a fuzzy i-open set.

iii) Since every fuzzy α-open set is a fuzzy i-open set.

Theorem.4.8. Every fuzzy i-contra-continuous is a fuzzy iα-contra-continuous.

proof. Let f: X→Y be a fuzzy i-contra-continuous and V and set in Y. Since, f is a fuzzy i-contra-continuous, then $f^{-1}(V)$ is a fuzzy i-closed set in X. Since, every fuzzy i-closed set is a fuzzy iα-closed, then $f^{-1}(V)$ is a fuzzy iα-closed set in X. Therefore, f is a fuzzy iα-contra-continuous.

Remark.4.4. The converses of Theorem.4.7. and Theorem.4.8. need not true as shown in the following in examples.

Example.4.6. Let X={a, b, c}, Y={0.1, 0.5, 0.7} and A, B be fuzzy subset defined as follows:

A(a)=0.4 A(b)=0.6 A(c)=0.4

B(0.1)=0.6 B(0.5)=0.4 B(0.7)=0.6

Let $\tau=\{0_X, 1_X, A\}$, $\phi=\{0_Y, 1_Y, B\}$. Then the function f: X→Y defined by

f(a) = 0.1, f(b) = 0.5, f(c) = 0.7

is a fuzzy i-contra-continuous, but not fuzzy contra-continuous, fuzzy semi-contra-continuous and fuzzy α-contra-continuous.

Example.4.7. Let X={a, b}, Y={0.3, 0.7} and A, B be fuzzy subset defined as follows:

A(a)=0.2 A(b)=0.1

B(0.3)=0.7 B(0.7)=0.7

Let $\tau=\{0_X, 1_X, A\}$, $\phi=\{0_Y, 1_Y, B\}$. Then the function f: X→Y defined by

f(a) = 0.7, f(b) = 0.3

is a fuzzy iα-contra-continuous, but not fuzzy i-contra-continuous.

References

[1] Zadeh L. A., Fuzzy sets, In form control. 8(1965), 53-338.
[2] Chang C. L., Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968), 90-182.

[3] Lomen R., Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56(1976), 621- 633.

[4] Chankraborty M. K., Ahsanullah T. M. G., Fuzzy topology on fuzzy sets and tolerance topology, fuzzy sets and systems. 45(1991), 97-189.

[5] Pu P. M., Lin, Fuzzy topology II product and quotient spacea, J. Math. Anal. Appl.77(1980), 20-37.

[6] Azad K. K., On fuzzy semi-continuous, fuzzy almost-continuous and fuzzy weakly-continuous, J. Math. Anal. Apple. 82(1981), 14-32.

[7] Malakar S., On fuzzy semi-irresolute and strongly irresolute function, Fuzzy sets and systems. 45(1992), 239-244.

[8] Erdal E., Etienne E. On fuzzy-contra-continuous, Advanced in fuzzy Math. 1(2006), 35-44.

[9] Singal M. K., Niti R. Fuzzy alpha-sets and alpha-continuous maps, J. Fuzzy sets and systems. 48(1992), 383-390.

[10] Din A. S., On fuzzy strong semi-continuity and pre-continuity, Fuzzy sets and systems. 44(1991), 3544.

[11] Wong C. K., Fuzzy points and local properties of fuzzy topology, Fuzzy sets and systems. 2(1987), 409-423.

[12] Parimala M., Devi R., On fuzzy contra α-irresolute maps, Fuzzy sets and systems. 5908(2009), 241-246.