Solving Nonlinear Partial Differential Equations Using Tanh and First Integral - Methods

حل المعادلات التفاضلية الجزئية غير الخطية باستخدام طريقتي الظل الزائدي و التكامل الأحادي

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Abstract

In this paper we will apply the Tanh-method and first integral method to find the exact solution of traveling wave nonlinear partial differential equations (PDEs.), These methods are powerful techniques to use symbolically compute traveling waves equations of nonlinear wave and evolution partial differential equations which a rising in a variety scientific fields , the methods provide a straightforward algorithm, two illustrating equations namely Fitzhugh-Nagumo and Burgers-Huxley are considered by each method in this study.

Keywords: Tanh method, First integral method, traveling wave equation.

الخلاصة

في هذا البحث سوف نطَبق طريقة – الظل الزائدي وطريقة التكامل الاحادي لإيجاد الحل التام لمعادلات تفاضلية جزئية غير خطية لانتقال الموجة . هذه الطرق ذات تقنية قوية في إستخدام الحسابات الرمزية لمعادلات الموجة من معادلات تفاضلية جزئية غير خطية والتي تظهر في مختلف الحقول العلمية . الطرق مزوَّدة بخوارزمية تقدمية ، معادلتين توضيحيتين تسَميان Fitzhugh-Nagumo و Burgers-Huxley تم حلهما بكل طريقة في هذه الدراسة.

1- Introduction

In recent years, directly searching for exact solutions of nonlinear partial differential equations (NPDEs.) has become more and more attractive partly due to the availability of computer symbolic systems like Mathematica which help us to find the solutions of system of nonlinear algebraic equations calculation on a computer, from the most effectively straightforward methods are Tanh method and the first integral method . The analytical method however are in most cases difficult to handle and require a thorough knowledge of its properties and possibilities before one is able to apply them to the problem . to solve nonlinear evolution and wave equations, one first starts to look for traveling waves solution, in particular, these waves can be found easily, because the PDE under consideration can immediately be transformed into ODE. In conservative systems, solutions are found by direct integration, suitable transformation or substitution or other techniques. These large class of wave phenomena plays a major role in science which frequently appear in different scientific domains and engineering fields, such as fluid dynamics [1], plasma physics, optical fibers [2], chemical kinetics involving reaction [3], mathematical biology (population dynamics) [4], solid state physics (lattice vibration) [5] , etc..

In this paper we present an adaptation of the Tanh method and first integral method to solve nonlinear PDEs. And to give an overview of possibilities these methods offers with some illustration examples, more details of basics can be found back in [6], for the sake of completeness we should mention that this technique is restricted to the search for stationary waves and we thus essentially deal with shock and/or solitary type of solutions. The useful Tanh-method is widely used by different authors such as [7,8,9,10,11,12,13,14,15,16,17,18,19,20,21] the method introduces a uniting method that one can find exact as well as approximate solutions in a straightforward and systematic way, the standard Tanh method depend on the balance method, where the linear terms of highest order are balanced with the highest order nonlinear terms of the reduced equations , the exact analytical solution for these nonlinear algebraic equation including the soliton solutions. We will introduce more details of Tanh method in section 2.

Authors in [22, 23, 24, 25, 26, 27, 28] introduced the first integral method for a reliable treatment of the NPDEs. We describe this method for finding the exact travelling wave solutions of nonlinear evolution equations in section 3 .The first integral method can be applied to models of various types of nonlinearity.

2- Outline of the Tanh Method

To avoid algebraic complexity, we introduced Tanh as a new variable, since all derivatives of a Tanh are represented by a Tanh it self .The main properties of the Tanh method will be explained, briefly discuss and then applied to particular, well-chosen examples, we refer to [11], [13], [22], [23] for more details. The nonlinear wave and
evolution equation (in principal one dimension) are commonly written as

$u_t= G(u,u_x ,u_{xx} ,\dots)$ or $u_{tt}= G(u,u_x ,u_{xx} ,\dots)$ …(2.1)

We like to know if these equations admit exact traveling wave solutions of (2.1) and how to compute them. The first step is to unite the independent variables x and t into a new variable through the definition $\xi = k(x - vt)$ which define the traveling frame of reference. Here (k >0) represents the wave number and v is the velocity of the traveling wave. Both are unknown parameters, we recall that the wave number k is inversely proportional to the width of the wave depending on the problem under study. This quantity will be determined or will remain a free and arbitrary parameter.

Typically for nonlinear waves, both velocity v as well as amplitude will be function of this parameter k. Accordingly, the dependent variable $u(x,t)$ is replaced by $U(\xi)$. Equations like eq. (2.1) are then transformed into

$$-kv\frac{dU}{d\xi} = G(U, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2},\dots) \text{ or } k^2v^2\frac{d^2U}{d\xi^2} = G(U, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2},\dots)\dots(2.2)$$

Hence, in the follows we deal with ODEs. rather than with PDEs.

to find the exact solutions for the ODEs. In tanh form, we introduce a new independent variable $Y = \tanh\xi$ into ODE. Where $U(\xi) = F(Y)$ this latter then depends solely on Y because $\frac{d}{d\xi}$ and subsequent derivatives in eq.(2.2) are now replaced by $\frac{d}{d\xi} = (1-Y^2)\frac{d}{dY}$,

$\frac{d^2}{d\xi^2} = -2(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2}$, etc…, therefore it make sense to attempt to find solutions we are looking for will be written as a finite power series in Y.

$$F(Y) = \sum_{n=0}^{N} a_n Y^n \quad \dots(2.3)$$

which incorporates solitary –wave and shock-wave profiles. Depending on the fact whether boundary conditions are involved or not. To determine N (highest order of Y) a balancing procedure is used , at most two terms proportional to Y^N must appear after substitution of eq.(2.2) into the equation under study . As a result of this analysis, we definitely require $a_{N+1} = 0$ and $a_N \neq 0$ for a particular N, in turns out that N =1 or 2 in most cases. This balance is obtained by comparing the behavior of Y^N in the highest derivative with the nonlinear term(s), (this is the most difficult step of the method in analyzing and solving the nonlinear algebraic system) with the help of Mathematica Package, as soon as N is determined in this way, we get after substitution of (2.3) into (2.2), (transformed to the Y variable). Algebraic equation for a_n (n=0, 1, 2…N). Depending on the problem under study. The wave number k will remain fixed or undetermined, as already mentioned, the velocity v of the travelling wave is always a function of k. If one is able to find nontrivial values for a_n(n = 0, 1, 2,…,N) in terms of known quantities a solution is ultimately obtained.

3-Outline of the First Integral – Method

Consider the general form of NPDE in the form P (u ,u_t, u_x ,u_{xx} ,u_{tt} …) = 0 … (3.1)

Using the wave variable $\xi = k(x - vt)$,so that $u(x,t) = U(\xi)$ eq.(3.1) transform into ODE of the form $Q(U,U',U'',U''') = 0$ ... (3.2)

Where the prime denotes the derivative with respect to ξ eq. (3.2) is then integrated as long as in all terms containing derivatives, the integration constants considered to be zeros, suppose that the solution of ODE (3.2) can be written as $u(x,t) = f(\xi)$,we introduce a new independent variables $X(\xi) = f(\xi)$, $Y(\xi) = f_{\xi}(\xi)$

which leads to the system of nonlinear ODEs.

$X_{\xi}(\xi) = Y(\xi)$, $Y_{\xi}(\xi) = F(X(\xi), Y(\xi))$ … (3.3)

According to the qualitative theory of ODEs.if we can find the integrals to eq.(3.3) then the general solutions of eq. (3.3) can be found directly . However , it is generally difficult to

find even one of the first integrals. Because there is not any systematic way to tell us how to find these integrals. So, we will use the division theorem to obtain one first integral of eq.(3.3) which reduces eq.(3.2) to first order integrable ODE. An exact solution to eq. (3.1) is then obtained by solving this equation. Now we come to recall the division theorem for two variables in the complex domain C.

Division Theorem:

Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$, and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that $Q(w,z) = P(w,z)\,G(w,z)$.

4- Exact Solutions of Equations by Using the Tanh Method:

Example 1: Consider the following Fitzhugh-Nagumo equation :

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - w(1-w)(a-w)\ldots(4.1)$$

We make the transformation w(x, t) = W (ξ) , $\xi = k(x - vt)$, eq.(4.1) becomes the ODE :

$$kvW' + k^2 W'' - aW + (1+a)W^2 - W^3 = 0\ldots(4.2)$$

We postulate the following Tanh series, and the transformation given, then eq.(4.2) reduces to :

$$kv(1-Y^2)\frac{dW}{dY} + k^2(-2Y(1-Y^2))\frac{dW}{dY} + (1-Y^2)^2$$ Balancing the power of linear term W'' and the

$$\frac{d^2W}{dY^2} - aW + (1+a)W^2 - W^3 = 0 \cdots\cdots\cdots (4.3)$$

nonlinear term W^3 $(Y^4 Y^{m-2} = Y^{3m})$ gives m+2=3m, and thus, m=1. The resulting ODE is then solved by a finite series of Tanh functions of the form

$$W(\xi) = \sum_{i=0}^{1} a_i Y^i = a_0 + a_1 Y,\ a_1 \neq 0 \cdots\cdots\cdots(4.4)$$

Where Y = Tanh(ξ) . Substituting eq. (4.4) in eq.(4.3) ,then equating the coefficient of Y^i , $i = 0,1,2,3$ leads to the following nonlinear system of algebraic equations :

$$\left.\begin{array}{l} Y^0:\ kva_1 - aa_0 + (1+a)a_0^2 - a_0^3 = 0 \\ Y^1:\ (-2k^2 - a) + 2(1+a)a_0 - 3a_0^2 \quad = 0 \\ Y^2:\ -kv \ + (1+a)a_1 - 3a_0a_1 \quad = 0 \\ Y^3: 2k^2 - a_1^2 = 0 \end{array}\right\} \cdots (4.5)$$

Solving the system of eq.(4.5) by by the help of Mathematica package gives:

$$a_0 = \tfrac{1}{2}\ , a_1 = \tfrac{-1}{2}\ ,\ k = \tfrac{-1}{2\sqrt{2}}\ ,\ v = \frac{\frac{-1}{\sqrt{2}} + 2\sqrt{2}\,a - \frac{5a^2}{\sqrt{2}} + \sqrt{2}\,a^3}{1-2a+a^2} \quad \cdots\cdots\cdots (4.6)$$

$$a_0 = \tfrac{1}{2}\ , a_1 = \tfrac{1}{2}\ ,\ k = \tfrac{-1}{2\sqrt{2}}\ ,\ v = \frac{\sqrt{2} - 4\sqrt{2}\,a + 5\sqrt{2}\,a^2 - 2\sqrt{2}\,a^3}{2(1-2a+a^2)} \quad \cdots (4.7)$$

$$a_0 = \tfrac{1}{2}\ , a_1 = \tfrac{-1}{2}\ ,\ k = \tfrac{1}{2\sqrt{2}}\ ,\ v = \frac{\sqrt{2} - 4\sqrt{2}\,a + 5\sqrt{2}\,a^2 - 2\sqrt{2}\,a^3}{2(1-2a+a^2)} \quad \cdots (4.8)$$

$$a_0 = \tfrac{1}{2}\ , a_1 = \tfrac{1}{2}\ ,\ k = \tfrac{1}{2\sqrt{2}}\ ,\ v = \frac{\frac{-1}{\sqrt{2}} + 2\sqrt{2}\,a - \frac{5a^2}{\sqrt{2}} + \sqrt{2}\,a^3}{1-2a+a^2} \quad \cdots (4.9)$$

$$a_0 = \tfrac{a}{2}\ , a_1 = \tfrac{-a}{2}\ ,\ k = \tfrac{-a}{2\sqrt{2}}\ ,\ v = \frac{2\sqrt{2} - 5\sqrt{2}\,a + 4\sqrt{2}\,a^2 - \sqrt{2}\,a^3}{2(1-2a+a^2)} \quad \cdots (4.10)$$

$$a_0 = \tfrac{a}{2}\ , a_1 = \tfrac{a}{2}\ ,\ k = \tfrac{-a}{2\sqrt{2}}\ ,\ v = \frac{-\sqrt{2} + \frac{5}{\sqrt{2}}\,a - 2\sqrt{2}\,a^2 + \frac{1}{\sqrt{2}}\,a^3}{1-2a+a^2} \quad (4.11)$$

$$a_0 = \tfrac{a}{2}\ , a_1 = \tfrac{-a}{2}\ ,\ k = \tfrac{a}{2\sqrt{2}}\ ,\ v = \frac{-\sqrt{2} + \frac{5}{\sqrt{2}}\,a - 2\sqrt{2}\,a^2 + \frac{1}{\sqrt{2}}\,a^3}{1-2a+a^2} \quad \cdots (4.12)$$

$$a_0 = \tfrac{a}{2}\ , a_1 = \tfrac{a}{2}\ ,\ k = \tfrac{a}{2\sqrt{2}}\ ,\ v = \frac{2\sqrt{2} - 5\sqrt{2}\,a + 4\sqrt{2}\,a^2 - \sqrt{2}\,a^3}{2(1-2a+a^2)} \quad \cdots (4.13)$$

$$a_0 = \tfrac{1+a}{2}\ , a_1 = \tfrac{1-a}{2}\ ,\ k = \tfrac{1-a}{2\sqrt{2}}\ ,\ v = \frac{-\sqrt{2} + \sqrt{2}\,a + \sqrt{2}\,a^2 - \sqrt{2}\,a^3}{2(1-2a+a^2)} \quad \cdots (4.14)$$

$$a_0 = \frac{1+a}{2} \ , a_1 = \frac{-1+a}{2} \ , \ k = \frac{1-a}{2\sqrt{2}} \ ,$$
$$v = \frac{\sqrt{2} - \sqrt{2}\,a - \sqrt{2}\,a^2 + \sqrt{2}\,a^3}{2(1-2a+a^2)} \cdots (4.15)$$

$$a_0 = \frac{1+a}{2} \ , a_1 = \frac{1-a}{2} \ , \ k = \frac{-1+a}{2\sqrt{2}} \ ,$$
$$v = \frac{\sqrt{2} - \sqrt{2}\,a - \sqrt{2}\,a^2 + \sqrt{2}\,a^3}{2(1-2a+a^2)} \quad (4.16)$$

$$a_0 = \frac{1+a}{2} \ , a_1 = \frac{-1+a}{2} \ , \ k = \frac{-1+a}{2\sqrt{2}} \ ,$$
$$v = \frac{-\sqrt{2} + \sqrt{2}\,a + \sqrt{2}\,a^2 - \sqrt{2}\,a^3}{2(1-2a+a^2)} \cdots (4.17)$$

Substituting the solutions set eqs. (4.6) - (4.17) and the corresponding solutions of eq. (4.4), we have the solutions of eq.(4.1) as follows :

$$W_1 = \tfrac{1}{2} - \tfrac{1}{2}\tanh(\tfrac{-1}{2\sqrt{2}}(x - vt)) \cdots (4.18)$$

$$W_2 = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\tfrac{-1}{2\sqrt{2}}(x - vt)) \cdots (4.19)$$

$$W_3 = \tfrac{1}{2} - \tfrac{1}{2}\tanh(\tfrac{1}{2\sqrt{2}}(x - vt)) \cdots (4.20)$$

$$W_4 = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\tfrac{1}{2\sqrt{2}}(x - vt)) \cdots (4.21)$$

$$W_5 = \tfrac{a}{2} - \tfrac{a}{2}\tanh(\tfrac{-a}{2\sqrt{2}}(x - vt)) \cdots (4.22)$$

$$W_6 = \tfrac{a}{2} + \tfrac{a}{2}\tanh(\tfrac{-a}{2\sqrt{2}}(x - vt)) \cdots (4.23)$$

$$W_7 = \tfrac{a}{2} - \tfrac{a}{2}\tanh(\tfrac{a}{2\sqrt{2}}(x - vt)) \cdots (4.24)$$

$$W_8 = \tfrac{a}{2} + \tfrac{a}{2}\tanh(\tfrac{a}{2\sqrt{2}}(x - vt)) \cdots (4.25)$$

$$W_9 = \tfrac{1+a}{2} + \tfrac{1-a}{2}\tanh(\tfrac{1-a}{2\sqrt{2}}(x - vt)) \cdots (4.26)$$

$$W_{10} = \tfrac{1+a}{2} + \tfrac{a-1}{2}\tanh(\tfrac{1-a}{2\sqrt{2}}(x - vt)) \cdots (4.27)$$

$$W_{11} = \tfrac{1+a}{2} + \tfrac{1-a}{2}\tanh(\tfrac{a-1}{2\sqrt{2}}(x - vt)) \cdots (4.28)$$

$$W_{12} = \tfrac{1+a}{2} + \tfrac{a-1}{2}\tanh(\tfrac{a-1}{2\sqrt{2}}(x - vt)) \cdots (4.29)$$

Figure (4.1) represents the solitary $w_{12}(x,t)$ in Eq. (4.29) with $v = 0.5$.

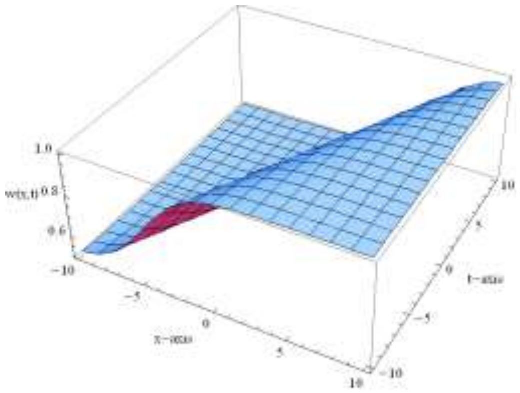

Figure (4.1): The Solitary Solution of Eq. (4.29) for $-10 \le t \le 10, -10 \le x \le 10$.

Example 2: Consider the following **Burgers – Huxley** equation:

$u_t = u_{xx} + uu_x + u(a-u)(u-1) \quad , a \neq 0 \cdots (4.30)$

We make the transformation w(x, t) =W (ξ) , $\xi = k(x - vt)$, eq.(4.30) becomes the ODE :

$k^2 u'' + ku'(v+u) - au + (a+1)u^2 - u^3 = 0 \cdots (4.31)$

We postulate the following Tanh series, and the transformation, then eq. (4.31) reduces to :

$$k^2[-2Y(1-Y^2)\tfrac{dU}{dY}] + k(v+U)(1-Y^2))\tfrac{dU}{dY} -$$
$$aU + (a+1)U^2 - U^3 = 0 \qquad \cdots (4.32)$$

Balancing the linear term U'' and the nonlinear term U^3 gives m+2=3m, and thus, m=1. The resulting ODE is then solved by a finite series of Tanh functions of the form

$W(\xi) = \sum_{i=0}^{1} a_i Y^i = a_0 + a_1 Y \;, a_1 \neq 0 \cdots (4.33)$ Where Y = Tanh (ξ) . Substituting eq. (4.33) in eq.(4.31) ,then equating the coefficient of

$Y^i \;, i = 0,1,2,3$ Leads to the following nonlinear system of algebraic equations:

$$\left.\begin{array}{l} Y^0 : \; ka_0a_1 + kva_1 - aa_0 + (1+a)a_0^2 - a_0^3 = 0 \\ Y^1 : \; (-2k^2 - a) + 2(1+a)a_0 - 3a_0^2 + ka_1 \quad = 0 \\ Y^2 : \; -kv \; - ka_0 + (1+a)a_1 - 3a_0a_1 \quad = 0 \\ Y^3 : 2k^2 - ka_1 - a_1^2 = 0 \end{array}\right\} \cdots (4.34)$$

Solving the system of eq.(4.34) by the help of Mathematica package gives:

$a_0 = \frac{1}{2} \;, a_1 = \frac{-1}{2} \;,\; k = \frac{-1}{2} \;,\; v = -1 + a \cdots (4.35)$

$a_0 = \frac{1}{2} \;, a_1 = \frac{1}{2} \;,\; k = \frac{-1}{4} \;,\; v = \frac{1}{2}(1-4a) \cdots (4.36)$

$a_0 = \frac{1}{2} \;, a_1 = \frac{-1}{2} \;,\; k = \frac{1}{4} \;,\; v = \frac{1}{2}(1-4a) \cdots (4.37)$

$a_0 = \frac{1}{2} \;, a_1 = \frac{1}{2} \;,\; k = \frac{1}{2} \;,\; v = -1 + a \cdots (4.38)$

$a_0 = \frac{a}{2} \;, a_1 = \frac{-a}{2} \;,\; k = \frac{-a}{2} \;,\; v = 1 - a \cdots (4.39)$

$a_0 = \frac{a}{2} \;, a_1 = \frac{a}{2} \;,\; k = \frac{-a}{4} \;,\; v = \frac{1}{2}(-4 + a) \cdots (4.40)$

$a_0 = \frac{a}{2}$, $a_1 = \frac{-a}{2}$, $k = \frac{a}{4}$, $v = \frac{1}{2}(-4+a) \cdots (4.41)$
$a_0 = \frac{a}{2}$, $a_1 = \frac{a}{2}$, $k = \frac{a}{2}$, $v = 1-a \cdots (4.42)$
$a_0 = \frac{1+a}{2}$, $a_1 = \frac{a-1}{2}$, $k = \frac{1-a}{4}$, $v = \frac{1+a}{2} \cdots (4.43)$
$a_0 = \frac{1+a}{2}$, $a_1 = \frac{1-a}{2}$, $k = \frac{1-a}{2}$, $v = -1-a \cdots (4.44)$
$a_0 = \frac{1+a}{2}$, $a_1 = \frac{1-a}{2}$, $k = \frac{-1+a}{4}$, $v = \frac{1+a}{2} \cdots (4.45)$

$a_0 = \frac{1+a}{2}$, $a_1 = \frac{-1+a}{2}$, $k = \frac{-1+a}{2}$, $v = -1-a \cdots (4.46)$

Substituting the solutions set eqs. (4.35)-(4.46) and the corresponding solutions of eq.(4.33) , we have the solutions of eq.(4.30) as follows :

$W_1 = \frac{1}{2} - \frac{1}{2}\tanh(\frac{-1}{2}(x - vt)) \quad \cdots (4.47)$
$W_2 = \frac{1}{2} + \frac{1}{2}\tanh(\frac{-1}{4}(x - vt)) \quad \cdots (4.48)$
$W_3 = \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{4}(x - vt)) \quad \cdots (4.49)$
$W_4 = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}(x - vt)) \quad \cdots (4.50)$
$W_5 = \frac{a}{2} - \frac{a}{2}\tanh(\frac{-a}{2}(x - vt)) \quad \cdots (4.51)$
$W_6 = \frac{a}{2} + \frac{a}{2}\tanh(\frac{-a}{4}(x - vt)) \quad \cdots (4.52)$
$W_7 = \frac{a}{2} - \frac{a}{2}\tanh(\frac{a}{4}(x - vt)) \quad \cdots (4.53)$
$W_8 = \frac{a}{2} + \frac{a}{2}\tanh(\frac{a}{2}(x - vt)) \quad \cdots (4.54)$
$W_9 = \frac{1+a}{2} + \frac{-1+a}{2}\tanh(\frac{1-a}{4}(x - vt)) \quad \cdots (4.55)$
$W_{10} = \frac{1+a}{2} + \frac{1-a}{2}\tanh(\frac{1-a}{2}(x - vt)) \quad \cdots (4.56)$

$W_{11} = \frac{1+a}{2} + \frac{1-a}{2}\tanh(\frac{a-1}{4}(x - vt)) \quad \cdots (4.57)$
$W_{12} = \frac{1+a}{2} + \frac{a-1}{2}\tanh(\frac{a-1}{2}(x - vt)) \quad \cdots (4.58)$

Figure (4.2) represents the solitary $w_1(x,t)$ in Eq.(4.47) with $v = 0.5$.

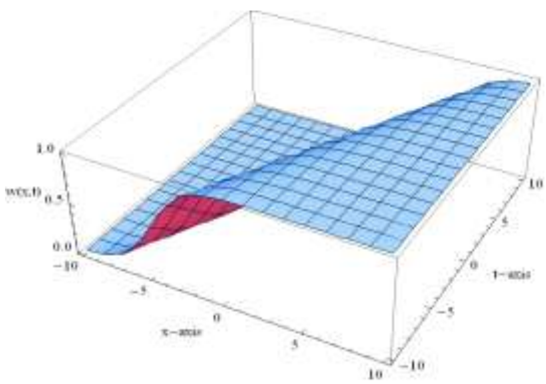

Figure (4.2): The Solitary Solution of Eq.(4.47) for $-10 \le t \le 10$,$-10 \le x \le 10$.

5- Exact Solutions Using the First Integral Method:

Example 1: Consider the following **Fitzhugh- Nagumo** equation:

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial t^2}-w(1-w)(a-w) \quad \cdots(5..1)$$

We make the transformation w(x,t)=W(ξ) , $\xi=k(x-vt)$, eq.(5.1) becomes the ODE :

$$kvW'+k^2W''-aW+(1+a)W^2-W^3=0\cdots(5.2)$$

Where the prime denote the derivation with respect to ξ .Using eq. (5.2) we obtain

$$X'(\xi)=Y(\xi) \quad \cdots\cdots\cdots\cdots\cdots\cdots(5.3)$$

$$Y'(\xi)=\tfrac{1}{k^2}[aX-(1+a)X^2+X^3-kvY]\cdots(5.4)$$

According to the first integral method, we suppose that $X(\xi)$ *and* $Y(\xi)$ are nontrivial solutions of eq. (5.3) and eq.(5.4), and $Q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i$ is irreducible polynomial in complex domain such that

$$Q(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X(\xi))Y^i(\xi)=0\cdots(5.5)$$

Where $a_i(X),\ (i=0,1,...,m)$ are polynomials of X *and* $a_m(X)\neq 0$. Eq. (5.5) is called first integral to eq. (5.3) and eq. (5.4) .According to the division theorem, there exists a polynomial $g(X)+h(X)Y$ in complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi}=\frac{\partial Q}{\partial X}\frac{\partial Q}{\partial Y}+\frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial \xi}=[g(X)+h(X)Y]\sum_{i=0}^{m}a_i(X)Y^i \cdots\cdots(5.6)$$

In this example, we assume that m=1 in eq.(5.5). Suppose that m=1, by equating the coefficients of $Y^i(i=0,1,2)$ on both sides of eq. (5.6), we have

$$a_1'(X)=a_1(X)h(X) \quad \cdots\cdots\cdots\cdots\cdots(5.7)$$

$$a_0'(X)-\tfrac{v}{k}a_1(X)=g(X)a_1(X)+a_0(X)h(X)\cdots(5.8)$$

$$g(X)a_0(X)=\frac{a_1(X)}{k^2}[aX-(1+a)X^2+X^3]\cdots(5.9)$$

Since $a_i(X)\ (i=0,1)$ are polynomials, then from eq.(5.5) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take

$a_1(X)=1$. Balancing the degrees of $g(X)$ *and* $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=AX+B \quad \cdots\cdots\cdots\cdots(5.10)$ Then we find $a_0(X)$,

$$a_0(X)=\tfrac{A}{2}X^2+(B+\tfrac{v}{k})X+C\cdots\cdots\cdots(5.11)$$

Where C is arbitrary integration constant.

Substituting $a_0(X)$ *and* $g(X)$ into eq.(5.9) and setting all the coefficients of power **X** to be zero, then we obtain a system of nonlinear algebraic equations, and using Mathematica solving them, we obtain

$$A=\tfrac{-\sqrt{2}}{k},\ B=0,\ C=\tfrac{-a}{k\sqrt{2}},\ v=\tfrac{1}{2}(\sqrt{2}+a\sqrt{2})\cdots(5.12)$$

$$A=\tfrac{-\sqrt{2}}{k},\ B=\tfrac{\sqrt{2}}{k},\ C=0,\ v=\tfrac{1}{2}(-2\sqrt{2}+a\sqrt{2})\cdots(5.13)$$

$$A=\tfrac{-\sqrt{2}}{k},\ B=\tfrac{a\sqrt{2}}{k},\ C=0,\ v=\tfrac{1}{2}(\sqrt{2}-2a\sqrt{2})\cdots(5.14)$$

$$A=\tfrac{\sqrt{2}}{k},\ B=0,\ C=\tfrac{a}{k\sqrt{2}},\ v=\tfrac{1}{2}(-\sqrt{2}-a\sqrt{2})\cdots(5.15)$$

$$A=\tfrac{\sqrt{2}}{k},\ B=\tfrac{-\sqrt{2}}{k},\ C=0,\ v=\tfrac{1}{2}(2\sqrt{2}-a\sqrt{2})\cdots(5.16)$$

$$A=\tfrac{\sqrt{2}}{k},\ B=\tfrac{-a\sqrt{2}}{k},\ C=0,\ v=\tfrac{1}{2}(-\sqrt{2}+2a\sqrt{2})\cdots(5.17)$$

Using eq. (5.12) into eq. (5.11) and eq. (5.5), we obtain

$$Y=\tfrac{\sqrt{2}}{2k}X^2-\tfrac{1}{2k}(\sqrt{2}+a\sqrt{2})X+\tfrac{a}{k\sqrt{2}}\cdots(5.18)$$

Combining eq. (5.18) with eq. (5.3), we obtain the exact solution of eq.(5.2) and then the exact solution of Fitzhugh-Nagumo equation (5.1) can be written as

$$w_1 = \frac{1 - a\, e^{(\frac{1+a}{k\sqrt{2}}\xi + \beta)}}{1 - e^{(\frac{1+a}{k\sqrt{2}}\xi + \beta)}} \cdots\cdots\cdots (5.19)$$

Where β is an arbitrary constant. Thus the travelling wave solution of Fitzhugh-Nagumo equation (5.1) can be written as

$$w_1 = \frac{1 - a\, e^{(\frac{1+a}{\sqrt{2}}(x - vt) + \beta)}}{1 - e^{(\frac{1+a}{\sqrt{2}}(x - vt) + \beta)}} \cdots\cdots\cdots (5.20)$$

Similarly, for the case of eqs . (5.13)- (5.17), the exact travelling wave solutions are:

$$w_2 = \frac{a}{1 - e^{(\frac{a}{\sqrt{2}}(x - vt) + \beta a)}} \cdots\cdots\cdots (5.21)$$

$$w_3 = \frac{1}{1 - e^{(\frac{1}{\sqrt{2}}(x - vt) + \beta)}} \cdots\cdots\cdots\cdots (5.22)$$

$$w_4 = \frac{a - e^{(\frac{1-a}{\sqrt{2}}(x - vt) + \beta)}}{1 - e^{(\frac{1-a}{\sqrt{2}}(x - vt) + \beta)}} \cdots\cdots\cdots\cdots (5.23)$$

$$w_5 = \frac{a}{1 - e^{(\frac{-a}{\sqrt{2}}(x - vt) + \beta a)}} \cdots\cdots\cdots\cdots (5.24)$$

$$w_6 = \frac{1}{1 - e^{(\frac{-1}{\sqrt{2}}(x - vt) + \beta)}} \cdots\cdots\cdots\cdots\cdots (5.25)$$

The following Figure (5.1) represents the solitary $w(x,t)$ in Eq.(5.23) with $v = 0.5, \beta = 0, a = 0.5.$

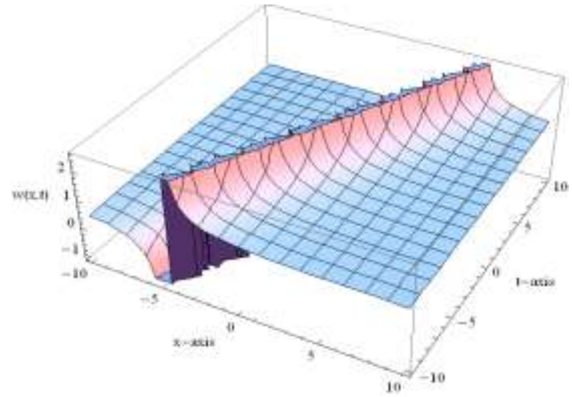

Figure (5.1): The Solitary Solution of Eq.(5.23) for $-10 \le t \le 10, -10 \le x \le 10.$

Example 2: Consider the following **Burgers- Huxley** equation :

$u_t = u_{xx} + uu_x + u(a-u)(u-1) \quad , a \ne 0 \cdots\cdots (5.26)$

We make the transformation w(x, t) =W (ξ) , $\xi = k(x - vt)$, eq.(5.26) becomes the ODE :

$k^2 u'' + ku'(v+u) - au + (a+1)u^2 - u^3 = 0 \cdots (5.27)$ Where the prime denote the derivation with respect to ξ . Using eq. (5.27) we obtain

$X'(\xi) = Y(\xi) \cdots\cdots\cdots\cdots\cdots\cdots (5.28)$

$Y'(\xi) = \frac{1}{k^2}[aX - (1+a)X^2 + X^3 - k(v+x)Y] \cdots (5.29)$ According to the first integral method, we suppose that $X(\xi)$ *and* $Y(\xi)$ are nontrivial solutions of eq.(5.28) and eq.(5.29), and $Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is irreducible

polynomial in complex domain $C[X,Y]$ such that

$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \cdots (5.30)$ Where $a_i(X)$, $(i = 0,1,...,m)$ are polynomials of X *and* $a_m(X) \ne 0$.

Eq. (5.30) is called first integral to eq.(5.28) and eq.(5.29) .According to the division theorem, there exists a polynomial $g(X) + h(X)Y$ in complex domain $C[X,Y]$ such that

$\frac{dQ}{d\xi} = \frac{\partial Q}{\partial X}\frac{\partial Q}{\partial Y} + \frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i \cdots\cdots\cdots (5.31)$ In this example, we assume that m=1 in eq.(5.30). Suppose that m=1, by equating the coefficients of $Y^i (i = 0,1,2)$ on both sides of eq.(5.31), we have

$a_1'(X) = a_1(X)h(X) \cdots\cdots\cdots\cdots\cdots\cdots (5.32)$

$a_0'(X) - \frac{(v+x)}{k} a_1(X) = g(X) a_1(X) + a_0(X) h(X) \cdots (5.33)$ $g(X) a_0(X) = \frac{a_1(X)}{k^2}[aX - (1+a)X^2 + X^3] \cdots (5.34)$ Since $a_i(X)\ (i = 0,1)$ are polynomials, then from eq.(5.30) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ *and* $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that

$g(X) = AX + B \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.35)$ Then we find $a_0(X)$,

$a_0(X) = \frac{1}{2}(A + \frac{1}{k})X^2 + (B + \frac{v}{k})X + C \cdots (5.36)$ Where C is arbitrary integration constant, substituting $a_0(X)$ *and* $g(X)$ into eq.(5.34) and setting all the coefficients of power **X** to be zero, then we obtain a system of nonlinear algebraic equations, and using Mathematica solving them, we obtain

$$A = \frac{-2}{k},\ B = 0,\ C = \frac{-a}{2k},\ v = \frac{1+a}{2} \cdots (5.37)$$
$$A = \frac{1}{k},\ B = 0,\ C = \frac{a}{k},\ v = -1-a \cdots (5.38)$$
$$A = \frac{1}{k},\ B = \frac{-1}{k},\ C = 0,\ v = 1-a \cdots (5.39)$$
$$A = \frac{-2}{k},\ B = \frac{2}{k},\ C = 0,\ v = \frac{-4+a}{2} \cdots (5.40)$$
$$A = \frac{1}{k},\ B = \frac{-a}{k},\ C = 0,\ v = -1+a \cdots (5.41)$$
$$A = \frac{-2}{k},\ B = \frac{2a}{k},\ C = 0,\ v = \frac{1-4a}{2} \cdots (5.42)$$

Using eq.(5.37) into eq.(5.36) and eq.(5.30) , we obtain

$$Y = \frac{1}{2k} X^2 - \frac{(1+a)}{2k} X + \frac{a}{2k} \qquad \ldots\ (5.43)$$

Combining eq.(5.43) with eq.(5.28), we obtain the exact solution of eq.(5.27) and then the exact solution of Burgers-Huxley eq. (5.26) can be written as

$$w_1 = \frac{a - e^{(a-1)(\frac{1}{2}\xi + \beta)}}{1 - e^{(a-1)(\frac{1}{2}\xi + \beta)}} \qquad \ldots.\ (5.44)$$

Where β is an arbitrary constant. Thus the travelling wave solution of Burgers-Huxley eq. (5.26) can be written as

$$w_1 = \frac{a - e^{(a-1)(\frac{1}{2}(x-vt) + \beta)}}{1 - e^{(a-1)(\frac{1}{2}(x-vt) + \beta)}} \qquad \ldots\ (5.45)$$

Similarly, for the case of eqs.(5.39)-(5.42), the exact travelling wave solutions are

$$w_2 = \frac{a\, e^{(a-1)(-(x-vt)+\beta)} - 1}{1 - e^{(a-1)(-(x-vt)+\beta)}} \qquad \ldots\ (5.46)$$

$$w_3 = \frac{-a\, e^{a(x-vt)+a\beta}}{1 - e^{a(x-vt)+a\beta}} \qquad \ldots\ (5.47)$$

$$w_4 = \frac{a}{1 - e^{\frac{a}{2}(x-vt)+a\beta}} \qquad \ldots\ (5.48)$$

$$w_5 = \frac{1}{1 - e^{-(x-vt)+\beta}} \qquad \ldots\ (5.49)$$

$$w_6 = \frac{1}{1 - e^{\frac{1}{2}(x-vt)+\beta}} \qquad \ldots\ (5.50)$$

The following Figure (5.2) represents the solitary $w(x,t)$ **in Eq.(5.49) with** $v = 0.5$, $\beta = 0$.

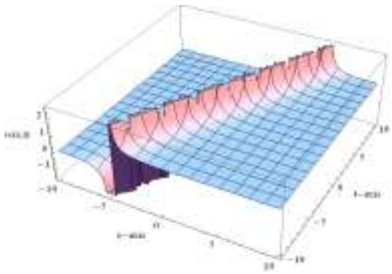

Fig.(5.2): The Solitary Solution of Eq.(5.49) for $-10 \le t \le 10$,$-10 \le x \le 10$.

6- Comparison of two Methods of Tanh and First Integral Method

In this section, we compare two methods of Tanh and first integral. We can use Tanh method to solve the nonlinear equations with any degree of derivative but when we can use the first integral method that the nonlinear equation is a second-order nonlinear differential equation or it can become to a second-order nonlinear differential equation, by using the first integral method we can obtain more solution of equation because with every step, we can put different m and find many solutions of the equation .But we obtain limited solutions of equation in Tanh method.

In both methods, because the calculations are done with aim of Mathematica, so we have no problem of difficulty in the calculations or time-consuming calculations for any of the two methods.

7-Conclusion

We presented two methods (Tanh method and First integral method) to compute special solution of nonlinear (PDEs.) without using explicit integration, it is shown with the aid of some examples that the methods in its present form is a powerful technique for investigating nonlinear wave equation, in particular those where diffusion is involved this technique is depend on the hypothesis of the travelling wave equations that expressed in terms of a Tanh .

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