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On Conjugate Space of 2-Fuzzy Generalized 2-Normed Space

حول الفضاء المرافق للفضاء 2-المعياري المعمم 2-الضبابي

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Abstract:

The main goal of this paper is to prove the extension theorem for 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional in 2-fuzzy generalized 2-normed space. Also, the definition of the 2-fuzzy adjoint operator of 2-fuzzy generalized 2-fuzzy bounded 2-fuzzy 2-linear operator defined on 2-fuzzy generalized 2-normed space is introduced.

الخلاصة:

ان الهدف الرئيسي من هذا البحث هو برهان مبرهنة التوسع للدالي 2-الخطي 2-الضبابي 2- المقيد 2-الضبابي المعمم 2-الضبابي في الفضاء 2- المعياري المعمم 2-الضبابي. ايضا قدم تعريف المؤثر المجاور2- الضبابي للمؤثر2-الخطي 2- الضبابي 2-المقيد 2-الضبابي المعمم 2-الضبابي المعرف على الفضاء 2-المعياري المعمم 2-الضبابي.

1. Introduction:

The theory of 2-norm on a linear space has introduced and developed by Gahler in [1]. In 2006 Lewandowska and et.al. [2] introduced the notation of Hahn-Banach extension theorem in generalized 2-normed space. Somasundaram and Beaula [3] defined the notion of 2-fuzzy 2-normed linear space. Later, Thangaraj and Angeline [4] introduced Hahn-Banach theorem in the realm of 2-fuzzy 2-normed linear spaces. Faria and Rasha [5], introduced and proved the form of Hahn-Banach theorem in generalized 2-normed spaces and gave the definition of adjoint operator for generalized 2-bounded 2-linear operator. In this paper we redefined in a general setting the idea of generalized 2-normed space, that appeared in ([2],[5]) and give 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator and studies extension theorem for 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy linear functional. Also, we give definition of 2-fuzzy adjoint operator of 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator. Moreover, we prove that the 2-fuzzy adjoint operator has the same norm as the 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator itself.

2. 2-Fuzzy Adjoint Operator of 2-Fuzzy Generalized 2-Fuzzy 2-Bounded 2-Fuzzy 2-Linear Operator:

In this section, we give a definition of 2-fuzzy generalized 2-normed space which based the idea that appeared in [2]. Also, some facts that appeared in [2] are generalized to 2-fuzzy setting.

Definition (2.1), [2]:

Let X be a real linear space of dimension greater than one. A function $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ is said to be a generalized 2-norm on X in case for each x, y and $z \in X$ and for each $\alpha \in R$,

$(N_1)\ \|x, \alpha y\| = |\alpha| \cdot \|x, y\| = \|\alpha x, y\|$

$(N_2)\ \|x, y+z\| \le \|x, y\| + \|x, z\|$

$(N_3)\ \|x+y, z\| \le \|x, z\| + \|y, z\|$

The pair $(X \times X, \|.,.\|)$ will be referred to as a generalized 2-normed space on $X \times X$.

Definition (2.2), [4]:

Let X be real linear space and F(X) be the set of all fuzzy sets on X. For $U,V \in F(X)$ and $k \in R$, define

$U+V=\{(x+y,\lambda\wedge\mu)\mid (x,\lambda)\in U,(y,\mu)\in V\}$ and

$kU=\{(kx,\lambda)\mid (x,\lambda)\in U\}$

Definition (2.3), [4]:

A fuzzy linear space $\tilde{X}=X\times(0,1]$ over the real field R where the addition and scalar multiplication operation on $\tilde{X}$ are defined by

$(x,\lambda)+(y,\mu)=(x+y,\lambda\wedge\mu)$

$k(x,\lambda)=(kx,\lambda)$

Is a fuzzy normed space in case for each $(x,\lambda)\in\tilde{X}$

(1) $\|(x,\lambda)\|=0$ if and only if $x=0,\ \lambda\in(0,1]$

(2) $\|k(x,\lambda)\|=|k|\|(x,\lambda)\|$ for each $(x,\lambda)\in\tilde{X}$ and for each $k\in R$

(3) $\|(x,\lambda)+(y,\mu)\|\le\|(x,\lambda\wedge\mu)\|+\|(y,\lambda\wedge\mu)\|$ for each $(x,\lambda),(y,\mu)\in\tilde{X}$

(4) $\left\|(x,\vee_t\lambda_t)\right\|=\wedge_t\|(x,\lambda_t)\|$ for each $\lambda_t\in(0,1]$.

Definition (2.4), [4]:

Let X be a real linear space and F(X) be the set of all fuzzy sets in X the addition and scalar multiplication are defined by

$f+g=\{(x+y,\lambda\wedge\mu)\mid (x,\lambda)\in f,\ (y,\mu)\in g\}$ and

$kf=\{(kx,\lambda)\mid (x,\lambda)\in f, k\in R\}$

Definition (2.5),[4]:

Let X be a real linear space. A function $\|.\|:F(X)\to[0,\infty)$ is said to be norm on a F(X) in case for each $f,f_1,f_2\in F(X)$ and $k\in R$, the following conditions hold

(1) $\|f\|=0$ if and only if $f=0$

(2) $\|kf\|=|k|\|f\|$

(3) $\|f_1+f_2\|\le\|f_1\|+\|f_2\|$

The pair $(F(X),\|.\|)$ will be refered to as a fuzzy normed space.

Definition (2.6):

A 2-fuzzy generalized 2-normed space is a generalized 2-normed space on $F(X)\times F(X)$.

In order to make definition (2.6) as clear as possible we will consider the following example.

Example (2.7):

Let $(F(X),\|.\|)$ be a fuzzy normed space. For each $f_1,f_2\in F(X)$ and $k\in R$ define

$\|f_1,f_2\|=\|f_1\|\|f_2\|$

It is easy to check that $(F(X)\times F(X),\|.,.\|)$ is a 2-fuzzy generalized 2-normed space.

Definition (2.8), [4]:

Let X and Y be real linear spaces. A function T from $F(X)\times F(X)$ into $F(Y)$ is said to be 2-fuzzy 2-linear operator in case satisfies the following conditions: for all $f_1,f_2,f_3,f_4\in F(X)$.

(1) $T(f_1+f_2,f_3+f_4)=T(f_1,f_3)+T(f_1,f_4)+T(f_2,f_3)+T(f_2,f_4)$.

(2) $T(\alpha f_1,\beta f_2)=\alpha.\beta\ T(f_1,f_2)$, for all scalars α,β .

Definition (2.9), [4]:
Let X be real linear space. A 2-fuzzy 2-linear functional is a real valued function on $F(X)\times F(X)$ satisfies the following conditions: for all $f_1,f_2,f_3,f_4 \in F(X)$.

(1) $T(f_1+f_2,f_3+f_4)=T(f_1,f_3)+T(f_1,f_4)+T(f_2,f_3)+T(f_2,f_4)$.

(2) $T(\alpha f_1,\beta f_2)=\alpha.\beta\ T(f_1,f_2)$, for all scalars α,β .

Definition (2.10):
Let ($F(X)\times F(X)$, $\|\cdot,\cdot\|$) be a 2-fuzzy generalized 2-normed space and $(F(Y),\|.\|)$ be a fuzzy normed space a 2-fuzzy 2-linear operator $T:(F(X)\times F(X),\|.,.\|)\to(F(Y),\|.\|)$ is said to be 2-fuzzy generalized 2-fuzzy 2-bounded in case there is a constant $k>0$ such that $\|T(f_1,f_2)\|\le k\|f_1,f_2\|$, for each $f_1,f_2\in F(X)$.

Remark (2.11)
Let $(F(X)\times F(X),\|.,.\|)$ be 2-fuzzy generalized 2-normed space and $(F(Y),\|.\|)$ be fuzzy normed space. We denote the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator from $(F(X)\times F(X),\|.,.\|)$ by B($F(X)\times F(X)$,F(Y)).

Proposition (2.12):
Let $T:(F(X)\times F(X),\|.,.\|)\to(F(Y),\|.\|)$ be a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator. Then
$\|T\|=\text{Inf}\left\{k:\|T(f_1,f_2)\|\le k\|f_1,f_2\|;(f_1,f_2)\in F(X)\times F(X)\right\}$ is the norm on the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator T.

Theorem (2.13):
Let $(F(X)\times F(X),\|.,.\|)$ be 2-fuzzy generalized 2-normed space and M be a linear subspace of $F(X)\times F(X)$. If T is a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional defined on M then T can be extended to a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional T_0 defined on the whole space $F(X)\times F(X)$ such that $\|T\|=\|T_0\|$.

Proof

If $M=F(X)\times F(X)$ or $\|T\|=0$ then take $T=T_0$. Otherwise without lose the generality assume that $\|T\|=1$ consider the family Ǎ of all possible extentions of T of norm one. Partially order Ǎ with $\le$ as follows given (G_1,L_1) , $(G_2,L_2)\in$ Ǎ, put $(G_1,L_1)\le(G_2,L_2)$ if and only if G_2 is an extension of G_1 that is $L_1\subseteq L_2$, $G_2(f_1,f_2)=G_1(f_1,f_2)$ for each $(f_1,f_2)\in L_1$ and $\|G_2\|=\|G_1\|$.

The family Ǎ is non-empty, because $(T,M)\in$ Ǎ. Let $\Im$ be chain of Ǎ. Define $\tilde{L}=\bigcup_{(G,L)\in\Im} L$. Clearly $\tilde{L}$ is a real linear subspace of $F(X)\times F(X)$ and contains M. Define $\tilde{G}:\tilde{L}\to R$ by $\tilde{G}(f_1,f_2)=G(f_1,f_2)$ where G is associated with some L, $(G,L)\in\Im$, which contains (f_1,f_2). Then $\tilde{G}$ is a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional on $\tilde{L}$ that is an extinction of every G and $\|\tilde{G}\|=1$. So the constructed pair $(\tilde{G},\tilde{L})$ is hence an upper bound for the chain $\Im$. By using Zorn's lemma there exists a maximal element $(T_0,L_m)\in$ Ǎ. To complete the proof it is enough to show that $L_m=F(X)\times F(X)$. Suppose by contrary that there exists (f_0,g_0) in $F(X)\times F(X)\setminus L_m$. Then consider the linear space

$L' = L_m + R(f_0, g_0) = \{(f + \alpha f_0, g + \mu g_0); (f,g) \in L_m\}$. Define

$T' : L' \to R$

$T'(f + \alpha f_0, g + \mu g_0) = T_0(f,g) + \alpha\mu\gamma$ where $(f,g) \in L_m$ and $\gamma \in R$ will be chosen in such away that $\|T'\| = 1$. But $\|T'\| = 1$ provided that

$\left|T_0(f,g) + \alpha\mu\gamma\right| \le \left\|f + \alpha f_0, g + \mu g_0\right\|$ ……..…………………(1)

For each $(f,g) \in M$ and $\gamma \in R$. Replace (f,g) by $(-\alpha f, -\mu g)$, and divide both sides of (1) by $|\alpha\mu|$. Then the requirement is that

$\left|T_0(f,g) - \gamma\right| \le \left\|f - f_0, g - g_0\right\|$ ……………………………..(2)

For each $(f,g) \in M$ and $\gamma \in R$. Since T_0 is 2-fuzzy 2-linear by choosing γ in such a way that $T_0(f,g) - \|f - f_0, g - g_0\| \le \gamma \le T_0(f,g) - \|f - f_0, g - g_0\|$. Then (2) and therefore (1) holds. So we have proved that $(T', L') \in$ Ǎ , $(T_0, L_m) \ne (T', L')$ and $(T_0, L) \le (T', L')$

which is a contradiction.

Theorem (2.14):

Let (f_0, g_0) be a vector in the 2-fuzzy generalized 2-normed space $(F(X) \times F(X), \|\cdot,\cdot\|)$ such that $\|f_0, g_0\| \ne 0$. Then there exists a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional T_0, defined on the whole space, such that $T_0(f_0, g_0) = \|f_0, g_0\|$ and $\|T_0\| = 1$.

Proof:

Consider the linear space

$M = \{(\alpha f_0, \mu g_0)\}$ and consider the functional T, defined on M as follows $T(\alpha f_0, \mu g_0) = \alpha\mu\|f_0, g_0\|$

Clearly, T is a 2-fuzzy 2-linear functional with the property that $T(f_0, g_0) = \|f_0, g_0\|$.

Further, since for any $(f,g) \in M$

$\left|T(f,g)\right| = |\alpha\mu| \left\|f_0, g_0\right\| = \left\|\alpha f_0, \mu g_0\right\| = \|f,g\|$

We see that T is a 2-fuzzy 2-bounded 2-fuzzy 2-functional. Moreover $\|T\| = 1$. It now remains only to apply theorem to assert the existence of a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional defined on the whole space, extending T and having the same norm as T, that $\|T_0\| = 1$.

Notation (2.15):

Let us denote the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional defined on 2-fuzzy generalized 2-normed space $(F(X) \times F(X), \|.,.\|)$ by $(F(X) \times F(X))^*$ and we call conjugate space of 2-fuzzy generalized 2-normed space, and the set of all bounded linear functional defined on $F(Y)$ by $F(Y)^*$.

Proposition (2.16):-

Let $(F(X) \times F(X), \|.,.\|)$ be 2-fuzzy generalized 2-normed space. Then $((F(X) \times F(X))^*, \|.\|)$ is a complete normed linear space with norm defined by

$\|T\| = \text{Inf}\{k : |T(f_1, f_2)| \le k\|f_1, f_2\| : (f_1, f_2) \in F(X) \times F(X)\}$.

Proof:-

It is easy to see $((F(X)\times F(X))^*,\|.\|)$ is a normed linear space. In order to prove $(F(X)\times F(X))^*$ complete let $\{T_k\}_{k\in N}$ be a Cauchy sequence in $(F(X)\times F(X))^*$ thus $\lim_{k\to\infty}\|T_k - T_{k+p}\| = 0, \forall\ p = 1,2,\ldots$. Also,

$|(T_k - T_{k+p})(f_1,f_2)| \le \|T_k - T_{k+p}\|\|f_1,f_2\|$.Then

$|(T_k - T_{k+p})(f_1,f_2)| \longrightarrow 0$ as $k\longrightarrow\infty, \forall f_1,f_2 \in F(X)$. Thus $\{T_k(f_1,f_2)\}$ is a Cauchy sequence in R. Since $(R,\|.\|)$ is complete then $\lim_{k\to\infty} T_k(f_1,f_2) = y$ exists in $(R,\|.\|)$. Define $T: F(X)\times F(X)\to R$ by $T(f_1,f_2) = y$ then it can be easily verified that T is 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2- linear functional. Hence

$|T_k(f_1,f_2) - T_{k+p}(f_1,f_2)| \le \|T_k - T_{k+p}\|\|f_1,f_2\| \le \varepsilon\|f_1,f_2\|\ \forall k \ge N(\varepsilon),\ f_1,f_2\in F(X),\ p = 1,2,\ldots$. L

etting $p\longrightarrow\infty$ we get

$|T_k(f_1,f_2) - T(f_1,f_2)| \le \varepsilon\|f_1,f_2\|,\ \forall\ k\ge N(\varepsilon)$ and $\forall f_1,f_2 \in F(X)$. Thus

$\|T_k - T\| \le \varepsilon,\ \forall k \ge N(\varepsilon)$ .Then $\|T_k - T\|\longrightarrow 0$ as $k\longrightarrow\infty$.

Hence $(F(X)\times F(X))^*$ is complete.

Definition (2.17):

Let $T:(F(X)\times F(X),\|.,.\|)\to(F(Y),\|.\|)$ be a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator from a 2-fuzzy generalized 2-normed space $F(X)\times F(X)$ to a normed space F(Y). The operator $T^X: F(Y)^* \to (F(X)\times F(X))^*$ is defined by

$(T^X g)(f_1,f_2) = g(T(f_1,f_2)) = h(f_1,f_2),\ g\in F(Y),\ f_1,f_2\in F(X)$, is called the 2-fuzzy adjoint operator of T.

Next, we give the following theorem which is based on the idea that appeared in [5].

Theorem (2.18):

Let $(F(X)\times F(X), \|\cdot,\cdot\|)$ be a 2-fuzzy generalized 2-normed space and $(F(Y),\|.\|)$ be a normed space. If $T:(F(X)\times F(X),\|.,.\|)\to(F(Y),\|.\|)$ is a 2-fuzzy generalized 2-fuzzy 2-bouned 2-fuzzy 2-linear operator. Then $T^X: F(Y)^*\to(F(X)\times F(X))^*$ is a bounded linear operator and $\|T^x\| = \|T\|$.

Proof:-

Since the operator T^x with its domain $F(Y)^*$ is a linear space then

$$T^X(\alpha_1 g_1 + \alpha_2 g_2)(f_1,f_2) = (\alpha_1 g_1 + \alpha_2 g_2)T(f_1,f_2)$$
$$= \alpha_1 g_1 T(f_1,f_2) + \alpha_2 g_2 T(f_1,f_2)$$
$$= \alpha_1(T^X g_1)(f_1,f_2) + \alpha_2(T^X g_2)(f_1,f_2)$$

Also, $\|T^X g\| = \|h\| \le \|g\|\|T\|$

Moreover, $\|T^X\| = \inf\{K : \|T^X g\| \le K\|g\|\ \}$

Then, $\left\|T^{x}\right\| \leq \|T\|$

For every vector (f_1, f_2) in $F(X) \times F(X)$ such that $\|f_1, f_2\| \neq 0$, then there is $g_0 \in F(Y)^*$ such that $\|g_0\| = 1$ and $g_0(T(f_1, f_2)) = \|T(f_1, f_2)\|$.

Writing $h_0 = T^{X} g_0$, we obtain

$$\|T(f_1, f_2)\| = g_0(T(f_1, f_2)) = h_0(f_1, f_2) \leq \|h_0\|\|f_1, f_2\| = \left\|T^{X} g_0\right\|\|f_1, f_2\|$$

$$\leq \left\|T^{X}\right\|\|g_0\|\|f_1, f_2\|$$

since, $\|g_0\| = 1$, we have

$$\|T(f_1, f_2)\| \leq \left\|T^{X}\right\|\|f_1, f_2\|$$

But, $\|T(f_1, f_2)\| \leq \|T\|\|f_1, f_2\|$

where, $k = \|T\|$ is the smallest constant k such that $\|T(f_1, f_2)\| \leq k\|f_1, f_2\|$

Hence, $\left\|T^{x}\right\| \geq \|T\|$. Therefore $\left\|T^{x}\right\| = \|T\|$.

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