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HK-Space and Separation axioms of Spectra of maximal Sub modules

فضاء HK وبديهيات الفصل لفضاء الموديولات الجزئية العظمى

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البحث مستل

Abstract:

The main purpose of this research, is to study the relationship between some the topological properties of the space Max(M) and the algebraic properties of an R-module M, and vice versa.

We prove in this paper that if M is a multiplication R-module then M is semi local if and only if max(M) is an HK space. Also in our study we shall prove if M is a multiplication R-module and max(M) is an HK space then M is cyclic and we study Separation Axioms of the space maximal (M) which that we get the following results , If M is an M-top R-module and semi local then maximal (M) is T0 , T1 , T2 , T3 , T4 .

الخلاصة :

الهدف الرئيسي من هذا البحث هو دراسة العلاقة بين بعض الصفات التبلوجية للفضاء max(M) والصفات الجبرية للموديول M وبالعكس .

برهنا في هذا البحث انه اذا كان R M– موديولا جدائياً فأن M موديول شبه محلي اذا وفقط اذا كان max(M) هو فضاء HK .وكذلك سنبرهن في الدراسة بأنه اذا كان R M– موديولا جدائياً وكان الفضاء max(M) فضاء HK فأن M يكون دائرياً . ودرسنا بديهيات الفصل للفضاء max(M) والتي من خلالها حصلنا على النتائج التالية انه اذا كان M-top هو R– موديول وشبه محلي فأن الفضاء max(M) يكون T0 ، T1 ، T2 ، T3 ، T4 .

0.Introduction

A proper submodule N of an R module M is said to be prime if $xm\in N$ for $x\in R$ and $m\in M$ imply that either $m\in N$ or $x\in(N:M)$ [1]. The set of all prime submodules of M is called the spectrum of M and denoted by specM [1], and N is said to be a maximal submodule if does not exist as a submodule K of M such that $K\neq M$, $K\neq N$ and $N\subset K$ [2].The set of all maximal submodule of M is called the maximal spectrum of M, and denoted by Max(M) .

R. L. Mc Casland , M. E. Moore and P. F. Smith in [1] defined a topology on specM for a certain class of modules in the following way : for any submodule N of an R module M , let V(N) be the set of all prime submodules of M containing N and denote the collection of all subsets V(N) of specM by λ(M) . Then it can be shown that λ(M) contains the empty set and specM .Moreover , λ(M) is closed under arbitrary intersection .If λ(M) is closed under finite union , then the topology induced by λ(M) is called the Zariski topology on specM and in this case M is called a top module .

In this research, we will conduct similar work on Max(M) compared to mention specM.

1.Basic concepts

In this section we introduced definitions, propositions and theorems that are included throughout the work.

Definition (1.1) :-

For any submodule N of an R-module M, the set $\{P\in Max(M)|N\subseteq P\}$ is called 'the variety of N ' and denoted by V(N) .

Definition (1.2) :-

The collection of all subset V(N) of Max(M) will be denoted by λ (M) ; i.e., λ(M)={V(N) | N is a submodule of M } .

Proposition (1.3) :-

1. $V(0)=Max(M)$, $V(M)=\Phi$.
2. For any family of submodules $\{N_\alpha\}_{\alpha\in I}$ of M $\bigcap_{\alpha\in I} V(N_\alpha) = V(\sum_{\alpha\in I} N_\alpha)$.

Proof :-

1- Since $0 \subseteq P$ for every maximal submodule P , then V(0)=Max(M).
Let $V(M) \neq \Phi$; i.e. , there exists a maximal submodule q of M such that $M \subseteq q$ .Thus M=q that's a contradiction . Thus $V(M) = \Phi$.

2- Let $P \in \bigcap_{\alpha \in I} V(N_\alpha)$,thus $P \in V(N_\alpha)$ for every $\alpha \in I$, that is $N_\alpha \subseteq P$ for every $\alpha \in I$ .Thus $\sum_{\alpha \in I} N_\alpha \subseteq P$, that is $\bigcap_{\alpha \in I} V(N_\alpha) \subseteq V(\sum_{\alpha \in I} N_\alpha)$.

Let $p \in V(\sum_{\alpha \in 1} N_\alpha)$, thus $\sum_{\alpha \in 1} N_\alpha \subseteq P$. ■

Remark (1.4) :-

From above Proposition it is noted that λ(M) contains the empty set and Max(M) , and λ(M) is closed under arbitrary intersection.

Definition (1.5) :-

An R-Module M is called an M-top R-Module if λ(M) is closed under finite unions.

Example (1.6) :-

Take Z_4 as Z-Module is a local Module thus Z_4 is an M-top Module.

Example (1.7) :-

Let M= $Z_2 \oplus Z_2$ as Z-module, then the submodules of M are O=<(0,0)> , <(1,1)> , $Z_2 \oplus \{0\}$, $\{0\} \oplus Z_2$ and M .

$Max(M) = \{<(1,1)> , Z_2 \oplus \{0\} , \{0\} \oplus Z_2 \}$.

$V(Z_2 \oplus \{0\}) = Z_2 \oplus \{0\}$, $V(\{0\} \oplus Z_2) = \{0\} \oplus Z_2$.

$\lambda(M) = \{\Phi , Max(M) , \{Z_2 \oplus \{0\}\} , \{\{0\} \oplus Z_2\} , \{<(1,1)>\}\}$.

note that $V(Z_2 \oplus \{0\}) \cup V(\{0\} \oplus Z_2) = \{Z_2 \oplus \{0\}\} \cup \{\{0\} \oplus Z_2\} = \{Z_2 \oplus \{0\}, \{0\} \oplus Z_2\}$.

But does not exist a submodule J of M such that $\{Z_2 \oplus \{0\}, \{0\} \oplus Z_2\} = V(J)$

Thus , M= $Z_2 \oplus Z_2$ as Z-module is not an M-top R-module .That is , not every R-module is an M-top module .

From Example(1.7) ,it is noted that the Zariski topology on Max(M) does not always exist . Later , the condition which makes an R-module M is an M-top R-module will be given . Before giving this condition ,some needed definitions will be given:

Definition (1.8) :-

A submodule S of an R-module M will be called semi maximal if S is an intersection of maximal submodules .

Definition (1.9) :- [2]

A maximal submodule K of M will be called extraordinary if whenever N and L are semi maximal submodules of M with $N \cap L \subseteq K$ then $N \subseteq K$ or $L \subseteq K$.

Definition (1.10) :- [3]

For any proper submodule N of R-module M , the set of the intersection of all maximal submodule containing N is called the 'Jacobson radical of N' and denoted by J(N) .

Proposition (1.11) :-

1- V(N)=V(J(N)) .
2- V(N)=V(K) if and only if J(N)=J(K)

Proof :-

1- Since $N \subseteq J(N)$, then $V(J(N)) \subseteq V(N)$. Let $P \in V(N)$ .Thus , $N \subseteq P$.
Since $J(N) = \cap \{q \mid q$ is a maximal submodule of M containing N $\}$,then $J(N) \subseteq P$. That is $P \in V(J(N))$. Thus $V(N) \subseteq V(J(N))$ .Thus , V(N)=V(J(N)) .

2- In this way , it would be clear .■

Theorem (1.12) :-

The following statements are equivalent for an R-module M .

1- M is an M-top module .
2- Every maximal submodule of M is extraordinary .
3- $V(N)\bigcup V(L)=V(N\bigcap L)$ for any semi maximal submodules N and L of M .

Proof :-

(1→2) Let K be any maximal submodule of M and let N and L be semi maximal submodules of M such that $N\bigcap L \subseteq K$, by hypothesis , there exists a submodule J of M such that $V(N)\bigcup V(L)=V(J)$ Now , $N=\bigcap_{i\in I} K_i$, for some collection of maximal submodules K_i ($i\in I$).

For each $i\in I$, $K_i \in V(N)\subseteq V(J)$.

So that $J\subseteq K_i$, thus , $J \subseteq\bigcap_{i\in I} K_i=N$. Similarly $J\subseteq L$, thus $J\subseteq N\bigcap L$.

Now , $V(N)\bigcup V(L) \subseteq V(N\bigcap L) \subseteq V(J) = V(N)\bigcup V(L)$.

It follows that $V(N)\bigcup V(L)=V(N\bigcap L)$ .But $K\in V(N\bigcap L)$.

Now gives $K\in V(N)$ or $K\in V(L)$, i.e. , $N\subseteq K$ or $L\subseteq K$.

(2→3) Let G and H be semi maximal submodules of M clearly $V(G)\bigcup V(H) \subseteq V(G\bigcap H)$. Let $K\in V(G\bigcap H)$ then $G\bigcap H\subseteq K$ and hence $G\subseteq K$ or $H\subseteq K$, i.e., $K\in V(G)$or $K\in V(H)$.This proves that $V(G\bigcap H)\subseteq V(G)\bigcup V(H)$ and hence $V(G)\bigcup V(H) = V(G\bigcap H)$.

(3→1) Let S and T be any submodules of M . If V(S) is empty, then $V(S)\bigcup V(T) =V(T)$. Suppose that V(S) and V(T) are both non-empty , then $V(S)\bigcup V(T) = V(J(S))\bigcup V(J(T)) = V(J(S)\bigcap J(T))$, by (3) this proves(1) .■

Definition (1.13) :-

A topological space (X,T) is called T_1-space if for any two distinct points x1 and x2 of X , there are two open subsets Y1 ,Y2 of X such that $x1\in Y1$, $x2\notin Y1$ and $x2\in Y2$, $x1\notin Y2$, see[4] .

Proposition (1.14) :-[4]

A space (X,T) is a T_1-space if and only if every one point subset of X is closed .

Proposition (1.15) :-

Let M be any M-top R-module , then Max(M) is a T_1-space .

Proof :-

Since $V(P)=\{P\}$, for every $P\in Max(M)$ and that V(P) is a closed subset of Max(M) , then {P} is closed , for every $P\in Max(M)$. Thus , by proposition(1.14) , Max(M) is a T_1-space .■

Proposition (1.16) :-

Let M be any M-top R-module if M is semi-local , then Max(M) is the discrete space .

Proof :-

By proposition(1.15),Max(M) is T_1-space and since M is semi-local, thus Max(M) is a finite set . Thus , Max(M) is the discrete space .■

Definition (1.17) :-

An R-module M is Notherian (Artinian) if every ascending (descending) chain of submodules of M is finite ,[5] .

Corollary (1.18) :-

Let M be any M-top R-module if M is Notherian and Artinian , then Max(M) is the discrete space .

Proof :-Since M is Notherian and Artinian , M has a composition series.Thus, Max(M) is a finite set; that is , M is semi-local , thus by Proposition(1.16) Max(M) is the discrete space .■

Proposition (1.19) :-
Let M be an M-top R-module . If M is Notherian, then Max(M) is Notherian space .
Proof :-
Let $X_1 \subseteq X_2 \subseteq$ ….be ascending chain of open subsets of Max(M) . Thus, there exists a submodule N_i of M such that $X_i=X(N_i)$ for every i=1,2,… Hence ,$X(N_1) \subseteq X(N_2) \subseteq$…. Thus by Proposition(1.7) [2] , the ascending chain $J(N_1) \subseteq J(N_2) \subseteq$…. of submodules of M has been got , since M is Notherian , then there exists a positive integer n such that $J(N_n)=J(N_{n+1})$ =… .Hence by proposition[2],$X(N_n)=X(N_{n+1})$= …. This completes the proof .■

2. HK-Spaces in Maximal Spectra of M-Top Module

Recall that a space X will be called an HK-space if given any family $\{X_i \mid i \in I\}$of open subsets of X such that $\bigcap_{i\in I} X_i = \Phi$, then there exists a finite subset J of I such that $\bigcap_{i\in J} X_i = \Phi$, see[6] .

In this section , me studied the algebraic properties of the Top R-module M whenever the space Max(M) is an HK-space.

Definition (2.1) :- An R-module M is a multiplication module if for each submodule N of M , there exists an ideal A of R such that N=AM .

Proposition (2.2):-
Let M be a multiplication R-module , then Max(M) is an HK-space if and only if there exists $P_1, P_2 ,…, P_n \in$ Max(M) such that $J(0)=\bigcap_{i=1}^{n} P_i$.
Proof :-
→) Since Max(M) $=\bigcup V(P)$, $P \in$ Max(M) , then $\bigcap X(P)=\Phi$, $P \in$ Max(M) . Hence $P_1, P_2 ,…, P_n \in$ Max(M) such that $\bigcap_{i=1}^{n} X(P_i)=\Phi$ (Φ=X(0)) and since $\bigcap_{i=1}^{n} X(P_i)=X(\bigcap_{i=1}^{n} P_i)=\Phi$,[Theorem(1.5)] ,then $J(0)=J(\bigcap_{i=1}^{n} P_i)$ [Proposition[(1.4),2] . Now $\bigcap_{i=1}^{n} P_i \subseteq J(0) \subseteq \bigcap_{i=1}^{n} P_i$, then $J(0)=\bigcap_{i=1}^{n} P_i$.
←) Let $P_1, P_2 ,…, P_n \in$ Max(M) such that $J(0)=\bigcap_{i=1}^{n} P_i$ and let $\{N_i \mid i \in I\}$ be a family of submodules of M such that Max(M) $=\bigcup_{i\in I} V(N_i)$. By theorem(1.5) , $V(\bigcap_{i=1}^{n} P_i)=\bigcup_{i=1}^{n} V(P_i)$, then $\bigcup_{i=1}^{n} V(P_i)$= Max(M)= $\bigcup_{i\in I} V(N_i)$.Hence $\forall$ 1< i <n , $\exists$ N_i such that $P_i \in V(N_i)$, then $V(P_i) \subseteq V(N_i)$ .Now , Max(M) $=\bigcup_{i=1}^{n} V(P_i) \subseteq \bigcup_{i=1}^{n} V(N_i) \subseteq$ Max(M) .Therefore , Max(M)= $\bigcup_{i=1}^{n} V(N_i)$.■

Proposition (2.3):-
Let M be a multiplication R-module then Max(M) is an HK-space if and only if M is a semi-local module .
Proof :- →) Notice that $\forall$ $P \in$ Max(M) , {P} is a closed subset of Max(M) . So Max(M)=$\bigcup$ {P} , $P \in$ Max(M) , then $\exists$ $P_1, P_2 ,…, P_n \in$ Max(M) such that Max(M) $=\bigcup_{i=1}^{n} P_i$. Hence Max(M) ={ $P_1, P_2 ,…, P_n$} .
←) Let Max(M) ={ $P_1, P_2 ,…, P_n$} , then $J(0)=\bigcap_{i=1}^{n} P_i$. Hence , by Proposition(2.2) , Max(M) is an HK-space .■

Proposition (2.4):-

Let M be a multiplication R-module , then M is Notherian and Artinian if and only if Max(M) is an HK-space .

Proof :-

→) Since M is Notherian and Artinian , then M has a composition series by Corollary (1.18) . Thus , M is semi-local .Hence , Max(M) is an HK-space .

←) Let Max(M) is an HK-space , then by proposition(2.3) , M is semi-local . That is , M has a composition series, M is Notherian and Artinian .■

Definition (2.5) :- A topological space (X) is Artinian if every a descending chain of open subsets of X is finite ,[3] .

Corollary (2.6):-

Let M be a multiplication R-module .If Max(M) is an HK-space , then Max(M) is Notherian and Artinian space .

Proof :-

Let Max(M) be an HK-space , then by Proposition(2.4) M is Notherian and Artinian , thus by proposition(1.11) , Max(M) is Notherian and Artinian .■

Definition (2.7) :- A topological space (X,T) is called locally compact if and only if every point in X has a compact neighborhood ,[7] .

Definition (2.8) :- A subspace Y of a topological space (X,T) is compact if every open cover $\delta=\{U_i\in T| i\in I \}$ of Y has a finite subcover . If Y=X , then , X is called a compact space .(see[4]) .

Remark (2.9) :-

Every finite subspace is compact .

Proposition (2.10) :-[2]

Every compact space is locally compact.

Corollary (2.11):-

Let M be a multiplication R-module .If Max(M) is an HK-space , then there is the following cases :

1- Max(M) is the discrete space .
2- Max(M) is compact .
3- Max(M) is locally compact .

Proof :-

1- Since Max(M) is an HK-space . Then , M is semi-local by Proposition(2.3) . Thus , by Proposition(1.16) , Max(M) is the discrete space .
2- Since Max(M) is an HK-space . Then , M is semi-local by Proposition(2.3) . Thus , Max(M) is compact [by Remark (2.9)] .
3- Since Max(M) is compact. Thus , Max(M) is locally compact. [Proposition(2.10)] .■

Proposition (2.12):-[8]

Let M be a multiplication R-module and has a finite number of maximal submodules , then M is cyclic .

Proposition (2.13):-

Let M be a multiplication R-module .If Max(M) is an HK-space then M is a cyclic module .

Proof :-

Let Max(M) is an HK-space , then by Proposition(2.3) it has been stated that M is semi-local; that is , M has a finite number of maximal submodules . Thus , by [Proposition(2.12)] , M is a cyclic module .

The converse of this proposition is not true in general and the following example has been investigated .■

Example (2.14):-

The ring Z as a Z-module . Z is a cyclic module but Max(Z) is not HK-space , since Z is not semi-local module because Max(Z) is infinite .

Definition (2.15) :- An R-module M is called faithful if rM=0 then r=0 .

Proposition (2.16):-

Let M be a multiplication faithful finitely generated , then Max(M) is an HK-space if and only if Max(R) is an HK-space .

Proof :-

Since M be a multiplication faithful finitely generated R-module, then Max(M)≈ Max(R) .Thus , Max(M) is an HK-space if and only if Max(R) is an HK-space.■

3. Separation Properties of Max(M)

Definition (3.1) :- A space (X,T) is said to be a T_2-space if given any two distinct points x1 ,x2 $\in$ X , there are open subsets , U and V such that x1 $\in$ U , x2 $\in$ V and U $\bigcap$ V= Φ , see[4] .

Proposition (3.2) :- Let M be any M-top R-module , then Max(M) is a T_2-space if and only if for every two distinct maximal submodules P and q of M , there are two submodules N and K of M such that $N \not\subset P$, $K \not\subset q$ and J(N) $\bigcap$ J(K)=J(0) .

Proof :-

The proof is simple and hence is omitted .■

Proposition (3.3) :-

Let M be any M-top R-module if M is semi-local , then Max(M) is a T_2-space .

Proof :-

Since M is semi-local . Then by Proposition(1.16) it indicates that Max(M) is the discrete space . Thus , Max(M) is a T_2-space .[The discrete space is a T_2-space] .[2]■

Corollary (3.4) :-

Let M be any M-top R-module if M is Notherian and Artinian. Then Max(M) is a T_2-space.

Definition (1.3) :- A space (X,T) is said to be a T_3-space if given any closed subset F of X and any point x of X which is not in F , there are open subsets , U and V such that x $\in$ U , F $\subseteq$ V and U $\bigcap$ V= Φ , see[4] .

Proposition (3.5) :-

Let M be any M-top R-module , then Max(M) is a T_3-space if and only if for every maximal submodule P of M and any submodule N of M such that $N \not\subset P$,there exist two submodules K and L such that $L \not\subset P$, J(K) $\bigcap$ J(L)=J(0) and J(N+K)=M .

Proof :- The proof is simple and is left to the reader .■

Proposition (3.6) :-

Let M be any M-top R-module if M is semi-local , then Max(M) is a T_3-space .

Proof :-

Since M is semi-local ,then by Proposition(1.16) , it is indicated that Max(M) is the discrete space . Thus Max(M) is a T_3-space .■

Corollary (3.7) :- Let M be any M-top R-module if M is Notherian and Artinian. Then , Max(M) is a T_3-space.

Definition (3.8) :- A topological space (X,T) is said to be a T_4-space if given any two disjoint closed subsets F and F' of X there are disjoint open subsets U and V such that F $\subseteq$ U and F' $\subseteq$ V , see[4] .

Proposition (3.9) :-

Let M be any M-top R-module , then Max(M) is a T_4-space if and only if for every two submodules N and K of M such that J(N+K)=M there exist two submodules I and L of M such that J(N+I)=M , J(K+L)=M and J(I) $\bigcap$ J(L)=J(0) .

Proof :-

The proof is simple and hence is omitted .■

Proposition (3.10) :-

Let M be any M-top R-module if M is semi-local , then Max(M) is a T_4-space .

Corollary (3.11) :-

Let M be an M-top R-module if M is Notherian and Artinian. Then , Max(M) is a T_4-space.

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