Bayesian Estimation of the Parameter of the Exponential Distribution with Different Priors under Symmetric and Asymmetric Loss Functions

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Received on: 28/1/2013 & Accepted on: 4/7/2013

ABSTRACT

The objective of this study is to compare the performance of some Bayesian estimators for the shape parameter of the exponential distribution. We considered three priors: the extension of Jeffreys as non- informative prior information, as well as the inverted gamma conjugate prior and the inverted chi square prior as informative prior information's. Bayes estimators have been obtained under symmetric and asymmetric loss functions :the quadratic loss function QLF and the general entropy loss function GELF, which is a modified version of the linear exponential loss function loss function LINEX. The comparison of Bayes estimators was made through a Monte Carlo simulation study on the performance of these estimators with respect to the mean square error MSE as a measure of performance.

The results of comparison showed that Bayes estimators of the shape parameter under the GELF with proper choice of γ, is a suitable alternative to the QLF when the loss is asymmetric in nature. Comparison also show that the informative priors performed better than the non-informative prior. Accordingly; if adequate information is available about the parameters it is preferable to use conjugate informative priors, otherwise the extension of Jeffreys prior gives quite reasonable results.

Keywords: Exponential Distribution, Bayes Estimators, Non-Informative and Informative Priors, Conjugate Prior, Quadratic and General Entropy Loss Functions.

تقديرات بايز لمعلمة التوزيع الاسي باستخدام دوال خسارة متماثلة وغير متماثلة

الخلاصة

يهدف البحث الى مقارنة اداء مجموعة من مقدرات بيز لمعلمة الشكل في التوزيع الاسي. أخذنا بالأعتبار ثلاثة دوال أولية وهي بالتحديد: دالة جيفريز الاولية الموسعة كدالة لامعلوماتية فضلا عن دالتي معكوس كاما و معكوس مربع كاي المعلوماتيتين. تم استخراج مقدرات بيز وفقا لدالتي خسارة متماثلة و غير متماثلة وهما: دالة الخسارة التربيعية (QLF) ودالة خسارة اخرى تدعى GELF (General Entropy Loss Function) وهي صيغة معدلة من دالة الخسارة الاسية - الخطية (LINEX). تمت مقارنة مقدرات بايز من خلال اسلوب مونت كارلو للمحاكاة حول اداء هذه المقدرات باستخدام متوسط مربعات الخطأ (MSE) كمعيار للمقارنة.

https://doi.org/10.30684/etj.32.5B.11
2412-0758/University of Technology-Iraq, Baghdad, Iraq

أظهرت نتائج المقارنة أن مقدرات بيز لمعلمة الشكل المستندة على دالة خسارة (GELF) عند قيم معينة للمعلمة γ تعد بديلا مناسبا لدالة الخسارة التربيعية عندما تكون الخسارة ذات طبيعة غير متماثلة .أظهرت المقارنة كذلك أن الدوال الاولية المعلوماتية كان أداءها أفضل من الدالة الأولية اللامعلوماتية. بموجب ذلك, اذا ما توفرت لدينا معلومات عن التوزيع الاولي للمعلمة يكون من الافضل استخدام دوال اولية معلوماتية, وخلاف ذلك فإن دالة جيفريز الموسعة أعطت نتائج عالية الجودة.

INTRODUCTION

The exponential distribution is one of the most popular distributions of failure time and life testing and reliability theory. Specifically, lifetime in the engineering sciences was nearly always modeled by the exponential distribution [7].

The Baysian approach has a lot of advantages in comparison with the classical approach; it can utilize the information in a formal way, satisfies the axioms of coherence and utilize decision theory [9].It is well known that, for Bayes estimators, the performance depend on the form of the prior distribution and the loss function assumed [8].Al-Kutubi H. S. and Ibrahim, N. A. (2009) studied the extension of Jeffreys prior information with square error loss function in exponential distribution [1].Al-Omari, M. A. (2010) also applied the same extension on Weibull distribution[2].Dey, S. (2010) obtained Bayes estimators of the shape parameter for the generalized exponential distribution based on a class of non-informative priors under the assumption of quadratic loss function QLF, squared log- error loss function SLELF, and general entropy loss function GELF[4]. Yarmohammadi, M. and Pazira, H. (2010) obtained Bayesian and non-Bayesian estimators for the shape parameter, reliability and failure rate functions of the generalized exponential distribution. Those estimators are obtained using symmetric and asymmetric loss functions [12].Also Dey, S. and Maiti, S. (2011) derived Bayes estimators of the shape parameter of exponentiated family distribution by considering extension of Jeffreys non-informative prior as well as conjugate priors under different scale-invariant loss functions [5].A comparison of priors made by Taher, M. and Saleem, M. (2011) showed that the informative priors are more advantageous than the non-informative priors[9].

The main aim of this paper is to obtain Bayes estimators of the exponential distribution and compare their performances under quadratic loss function which is symmetric and the general entropy loss function which is asymmetric. Here we use the extension of Jeffreys as a non-informative prior, the inverted gamma as a natural conjugate prior, and the inverted Chi square which is another form of the inverted gamma distribution as an informative prior. Comparison was made through a Monte Carlo simulation study on the performance of these estimators. The results are summarized in tables and followed by the conclusions.

The Model

Let t_1, t_2, …, t_n be independent identically distributed lifetimes from exponential distribution with an unknown parameter θ. The probability density function is given by:[1]

$$f(t,\theta)=\frac{1}{\theta}exp\left[-\frac{t}{\theta}\right] \qquad 0\le t<\infty;\ \theta>0 \qquad \dots\ (1)$$

The cumulative distribution function (cdf) is given by:

$$F(t,\theta) = 1 - exp\left(-\frac{t}{\theta}\right)$$

The likelihood function for the exponential pdf is given by:

$$L(t_i;\theta) = \left(\frac{1}{\theta}\right)^n exp\left[-\frac{\sum_{i=1}^{n} t_i}{\theta}\right] \qquad \dots (2)$$

PRIOR DISTRIBUTIONS

The quality of Bayes estimators requires appropriate choice of priors for the parameter. If we have enough information about the parameter, then it is better to use the informative priors. Otherwise we may resort to use non-informative priors [4]. The parameters of the prior distribution are called hyper parameters. All the hyper parameters are assumed to be known and non-negative. In this paper we consider both types of priors: the extension of Jeffreys as a non- informative prior and the inverted gamma and the inverted chi square as informative priors.

1. The extension of Jeffreys prior proposed by Al-Kutubi H. S. and Ibrahim, N. A. (2009) [1] is given as

$$\pi_1(\theta) = k\,\frac{n^c}{\theta^{2c}}\ ;\ c > 0,\ \text{and } k \text{ is a constant.} \qquad \dots (3)$$

2. The inverted gamma prior distribution with hyper parameters α and β.

This conjugate prior distribution is the distribution of the reciprocal of a variable distributed according to the gamma distribution. It is given as [9]

$$\pi_2(\theta) = \frac{\beta^{\alpha}\ e^{-\frac{\beta}{\theta}}}{\Gamma(\alpha)\ \theta^{\alpha+1}}\ ;\ \alpha,\beta,\theta > 0. \qquad \dots (4)$$

3. The inverted Chi square prior distribution with hyper parameters α and β. This distribution is the distribution of a random variable whose reciprocal divided by its degrees of freedom is a Chi square distribution. It is given as [9]

$$\pi_3(\theta) = \frac{\beta^{\alpha/2}\ e^{-\frac{\beta}{2\theta}}}{\Gamma\left(\frac{\alpha}{2}\right)\ \theta^{\frac{\alpha}{2}+1}}\ ;\ \alpha,\beta,\theta > 0. \qquad \dots (5)$$

LOSS FUNCTIONS

A loss function is used to represent a penalty associated with each estimate. The loss should be zero if and only if $\hat{\theta} = \theta$. In most cases, the squared error loss function, which is symmetrical, is frequently used by researchers. Its use is very popular because of its mathematical simplicity. The symmetric nature of the squared error loss function gives equal weight to over and under estimation of the parameters [10]. However asymmetric loss functions have been shown to be functional. A useful asymmetric loss function is the linear-exponential (LINEX) loss function, introduced by Varian (1975) [11]. This loss function behaves linearly for large underestimation errors, in which the exponential term vanishes, and exponentially for large overestimation errors, in which case the exponential tern dominates. With the above priors, we applied the following two loss functions

1. The quadratic loss function (QLF)

It is given by: [4]

$L_1(\theta,\delta) = \left(\frac{\theta-\delta}{\theta}\right)^2$, Where δ is a decision rule to estimate *θ*.
For the posterior distribution $h(\theta|\boldsymbol{t})$, δ is to be chosen such that
$\int_0^\infty \left(\frac{\theta-\delta}{\theta}\right)^2 h(\theta|\boldsymbol{t})\ d\theta$ is minimum.
This can be equivalently written as
$\int_0^\infty (\theta-\delta)^2 q(\theta|\boldsymbol{t}) d\theta$, with $q(\theta|\boldsymbol{t}) = \frac{1}{\theta^2}\ h(\theta|\boldsymbol{t})$ is minimum. Here $\delta = E(\theta|\boldsymbol{t})$.
Bayes' estimator for the parameter θ of the exponential distribution under quadratic loss function may be given as

$$\hat{\theta} = \frac{\int_0^\infty \frac{1}{\theta} h(\theta|t)\ d\theta}{\int_0^\infty \frac{1}{\theta^2} h(\theta|t)\ d\theta} \qquad \ldots (6)$$

2- The general entropy loss function (GELF)

This loss function is a particular type of asymmetric loss functions which is proposed by Calabria and Pulcini (1996) [3]. It is a suitable alternative to the modified linear exponential (MLINEX) loss function and is given by: [4]

$$L_2(\theta,\delta) = w\left[\left(\frac{\delta}{\theta}\right)^\gamma - \gamma\ ln\left(\frac{\delta}{\theta}\right) - 1\right];\ \gamma \neq 0, \qquad w > 0$$

Whose minimum occurs at $\delta = \theta$. Without loss of generality we assume that $w = 1$.
Bayes' estimator for the parameter θ of the exponential distribution under general entropy loss function may be given as

$$\hat{\theta} = [E(\theta^{-\gamma}|\boldsymbol{t})]^{-\frac{1}{\gamma}} \qquad \ldots (7)$$

BAYESIAN ESTIMATION

To obtain Bayes estimators, we assume that θ is a real valued random variable with probability density function $\pi(\theta)$. The posterior distribution of θ is the conditional probability density function of θ given the data. In this section we consider Bayes estimation of the unknown parameter θ based on the above mentioned priors and loss functions.

i) Extension of Jeffreys' prior information, under quadratic loss function

Combining the prior distribution in (3) and the likelihood function (2), the posterior distribution for the parameter θ given the data ($t_1, t_2 \ldots t_n$) is derived as follows:

$$h(\theta\,|\,\mathbf{t}) = \frac{\prod_{i=1}^n f(t|\theta)\pi_1(\theta)}{\int_0^\infty \prod_{i=1}^n f(t|\theta)\pi_1(\theta) d\theta} = \frac{e^{-\frac{\sum_{i=1}^n t_i}{\theta}}\quad \theta^{-n-2c}}{\int_0^\infty\ e^{-\frac{\sum_{i=1}^n t_i}{\theta}}\quad \theta^{-n-2c}\ d\theta}$$

Let

$$y = \frac{\sum_{i=1}^n t_i}{\theta},$$

Then

$$h(\theta|\boldsymbol{t}) = \frac{\mathrm{y}^{\mathrm{n}+2\mathrm{c}}\quad \mathrm{e}^{-\mathrm{y}}}{-\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}\ \int_0^\infty \mathrm{y}^{\mathrm{n}+2\mathrm{c}-2}\ \mathrm{e}^{-\mathrm{y}}\ \mathrm{dy}}$$

And the posterior distribution become as follows:

$$h(\theta|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i\right)^{n+2c-1} e^{-\frac{\sum_{i=1}^{n} t_i}{\theta}}}{\theta^{n+2c}\,\Gamma(n+2c-1)} \qquad \ldots (8)$$

According to the quadratic loss function, the corresponding Bayes' estimator of the parameter θ is derived by substituting the posterior distribution (8) in the numerator of (6), as follows:

$$\int_0^\infty \frac{1}{\theta}\, h(\theta|t)\, d\theta = \frac{-\left(\sum_{i=1}^{n} t_i\right)^{n+2c-1}}{\Gamma(n+2c-1)} \int_0^\infty \theta^{-n-2c-1}\; e^{\frac{-\sum_{i=1}^{n} t_i}{\theta}}\, d\theta$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i}{\theta}$$

Then

$$\int_0^\infty \frac{1}{\theta}\, h(\theta|t)\, d\theta = \frac{-\left(\sum_{i=1}^{n} t_i\right)^{n+2c-1}}{\Gamma(n+2c-1)} \int_0^\infty e^{-y} \left(\frac{y}{\sum_{i=1}^{n} t_i}\right)^{n+2c+1} \frac{-\sum_{i=1}^{n} t_i}{y^2}\; dy$$

Then

$$\int_0^\infty \frac{1}{\theta}\, h(\theta|t)\, d\theta = \frac{\left(\sum_{i=1}^{n} t_i\right)^{-1}}{\Gamma(n+2c-1)} \int_0^\infty e^{-y}\; y^{n+2c-1}\, dy \;= \frac{\left(\sum_{i=1}^{n} t_i\right)^{-1} \Gamma(n+2c)}{\Gamma(n+2c-1)}$$

In the same manner the denominator of (6) become as follows

$$\int_0^\infty \frac{1}{\theta^2}\, h(\theta|t)\, d\theta = \frac{\left(\sum_{i=1}^{n} t_i\right)^{-2} \Gamma(n+2c+1)}{\Gamma(n+2c-1)}$$

Hence,

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n} t_i}{n+2c} \qquad \ldots (9)$$

ii) Extension of Jeffreys' prior information, under the general entropy loss function

According to the general entropy loss function, the corresponding Bayes' estimator for the parameter θ is derived by substituting the posterior distribution (8) in (7), as follows:

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \int_0^\infty \theta^{-\gamma}\, h(\theta|t)\, d\theta$$

$$= \frac{-\left(\sum_{i=1}^{n} t_i\right)^{n+2c-1}}{\Gamma(n+2c-1)} \int_0^\infty \theta^{-n-2c-\gamma}\; e^{\frac{-\sum_{i=1}^{n} t_i}{\theta}}\, d\theta$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i}{\theta}$$

Then

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i\right)^{n+2c-1}}{\Gamma(n+2c-1)} \int_0^\infty \left(\frac{\sum_{i=1}^{n} t_i}{y}\right)^{-n-2c-\gamma} e^{-y}\, \frac{-\sum_{i=1}^{n} t_i}{y^2}\; dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\ \right)^{-\gamma}}{\Gamma(n+2c-1)} \int_0^{\infty} e^{-y}\, y^{n+2c+\gamma-2}\, dy = \frac{\left(\sum_{i=1}^{n} t_i\ \right)^{-\gamma} \Gamma(n+2c+\gamma-1)}{\Gamma(n+2c-1)}$$

Hence,

$$[E(\theta^{-\gamma}|\boldsymbol{t})]^{-\frac{1}{\gamma}} = \left(\frac{\left(\sum_{i=1}^{n} t_i\ \right)^{-\gamma} \Gamma(n+2c+\gamma-1)}{\Gamma(n+2c-1)}\right)^{-\frac{1}{\gamma}}$$

$$\hat{\theta}_2 = \sum_{i=1}^{n} t_i \left(\frac{\Gamma(n+2c-1)}{\Gamma(n+2c+\gamma-1)}\right)^{\frac{1}{\gamma}} \qquad \ldots (10)$$

iii) Inverted Gamma prior information, under quadratic loss function

Combining the prior distribution in (4) and the likelihood function (2), the posterior distribution for the parameter θ given the data (t_1, $t_{2,}$…, tn) is derived as follows:

$$h(\theta\,|\,\mathbf{t}) = \frac{\prod_{i=1}^{n} f(t_i|\theta)\pi_2(\theta)}{\int_0^{\infty} \prod_{i=1}^{n} f(t_i|\theta)\pi_2(\theta)d\theta} = \frac{e^{-\frac{\left(\sum_{i=1}^{n} t_i\ +\beta\right)}{\theta}} \quad \theta^{-n-\alpha-1}}{\int_0^{\infty} e^{-\frac{\left(\sum_{i=1}^{n} t_i\ +\beta\right)}{\theta}} \quad \theta^{-n-\alpha-1} \quad d\theta}$$

Let $\quad y = \frac{\sum_{i=1}^{n} t_i\ +\beta}{\theta}$,

Then

$$h(\theta|\boldsymbol{t}) = \frac{\mathrm{y}^{\mathrm{n}+\alpha+1}\quad \mathrm{e}^{-\mathrm{y}}}{-(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}\ +\beta) \int_0^{\infty} \mathrm{y}^{\mathrm{n}+\alpha-1}\ \mathrm{e}^{-\mathrm{y}}\ \ \mathrm{dy}}$$

$$= \frac{-\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}\ +\beta\right)^{\mathrm{n}+\alpha} \quad \mathrm{e}^{-\frac{\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}\ +\beta\right)}{\theta}}}{\theta^{\mathrm{n}+\alpha+1}\ \Gamma(\mathrm{n}+\alpha)} \qquad \ldots (11)$$

Now, according to the quadratic loss function, the corresponding Bayes' estimator for the parameter θ is derived by substituting the posterior distribution (11) in the numerator of (6), as follows:

$$\int_0^{\infty} \frac{1}{\theta}\ h(\theta|t)\ d\theta = \frac{-(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i} + \beta)^{\mathrm{n}+\alpha}}{\Gamma(\mathrm{n}+\alpha)} \int_0^{\infty} \theta^{-n-\alpha-2} \quad e^{\frac{-(\sum_{i=1}^{n} t_i+\beta)}{\theta}}\ d\theta$$

Let

$$y = \frac{(\sum_{i=1}^{n} t_i+\beta)}{\theta}$$

Then

$$\int_0^{\infty} \frac{1}{\theta}\ h(\theta|t)\ d\theta = \frac{-(\sum_{i=1}^{n} t_i + \beta)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^{\infty} e^{-y} \left(\frac{\sum_{i=1}^{n} t_i + \beta}{y}\right)^{-n-\alpha-2} \frac{-(\sum_{i=1}^{n} t_i + \beta)}{y^2}\, dy$$

$$= \frac{(\sum_{i=1}^{n} t_i + \beta)^{-1} \Gamma(n+\alpha+1)}{\Gamma(n+\alpha)}$$

In the same manner the denominator of (6) become as follows

$$\int_0^{\infty} \frac{1}{\theta^2}\ h(\theta|t)\ d\theta = \frac{(\sum_{i=1}^{n} t_i + \beta)^{-2} \Gamma(n+\alpha+2)}{\Gamma(n+\alpha)}$$

Hence,

$$\hat{\theta}_3 = \frac{\sum_{i=1}^{n} t_i + \beta}{n+\alpha+1} \qquad \dots (12)$$

iv) Inverted Gamma prior information, under the general entropy loss function

According to the general entropy loss function, the corresponding Bayes' estimator for the parameter θ is derived by substituting the posterior distribution (11) in (7), as follows:

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i \ +\beta\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{-n-\alpha-\gamma-1} \ e^{\frac{-\left(\sum_{i=1}^{n} t_i \ +\beta\right)}{\theta}} d\theta$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i + \frac{\beta}{2}}{\theta}$$

Then

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i \ +\beta\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \left(\frac{\sum_{i=1}^{n} t_i + \beta}{y}\right)^{-n-\alpha-\gamma-1} e^{-y} \frac{-\left(\sum_{i=1}^{n} t_i \ +\beta\right)}{y^2} dy$$

Hence,

$$E[(\theta|\boldsymbol{t})^{-\gamma}]^{-\frac{1}{\gamma}} = \left[\frac{\left(\sum_{i=1}^{n} t_i \ +\beta\right)^{-\gamma} \Gamma(n+\alpha+\gamma)}{\Gamma(n+\alpha)}\right]^{-\frac{1}{\gamma}}$$

Hence,

$$\hat{\theta}_4 = \left(\sum_{i=1}^{n} t_i + \beta\right) \left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+\gamma)}\right)^{\frac{1}{\gamma}} \qquad \dots (13)$$

It is interesting to note that equation (13) is a special case of equation (10) if we set $\beta = 0$, $\alpha = 2c\text{-}1$, and $\gamma = 1$.

v) Inverted Chi square prior information, under quadratic loss function

Combining the prior distribution in (5) and the likelihood function (2), the posterior distribution for the parameter θ given the data (t_1, $t_2 \dots t_n$) is derived as follows:

$$h(\theta \,|\, \mathbf{t}) = \frac{\prod_{i=1}^{n} f(t_i|\theta)\pi_3(\theta)}{\int_0^\infty \prod_{i=1}^{n} f(t_i|\theta)\pi_3(\theta) d\theta} = \frac{e^{-\frac{\left(\sum_{i=1}^{n} t_i \ +\frac{\beta}{2}\right)}{\theta}} \ \theta^{\frac{-2n-\alpha-2}{2}}}{\int_0^\infty e^{-\frac{\left(\sum_{i=1}^{n} t_i \ +\frac{\beta}{2}\right)}{\theta}} \ \theta^{\frac{-2n-\alpha-2}{2}} d\theta}$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i \ +\frac{\beta}{2}}{\theta},$$

then

$$h(\theta|\boldsymbol{t}) = \frac{y^{\frac{2n+\alpha+2}{2}} \; e^{-y}}{-(\sum_{i=1}^{n} t_i \; + \frac{\beta}{2}) \int_0^\infty y^{\frac{2n+\alpha-2}{2}} \; e^{-y} \; dy}$$

$$= \frac{-\left(\sum_{i=1}^{n} t_i \; + \frac{\beta}{2}\right)^{\frac{2n+\alpha}{2}} \; e^{-\frac{\left(\sum_{i=1}^{n} t_i \; + \frac{\beta}{2}\right)}{\theta}}}{\theta^{\frac{2n+\alpha+2}{2}} \; \Gamma\left(\frac{2n+\alpha}{2}\right)} \qquad \ldots (14)$$

Now, according to the quadratic loss function, the corresponding Bayes' estimator for the parameter θ is derived by substituting the posterior distribution (14) in the numerator of (6), as follows:

$$\int_0^\infty \frac{1}{\theta} \; h(\theta|t) \; d\theta = \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \int_0^\infty \theta^{\frac{-2n-\alpha-4}{2}} \;\; e^{\frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)}{\theta}} \; d\theta$$

Let

$$y = \frac{\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)}{\theta}$$

Then

$$\int_0^\infty \frac{1}{\theta} \; h(\theta|t) \; d\theta$$

$$= \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \int_0^\infty e^{-y} \left(\frac{\sum_{i=1}^{n} t_i + \frac{\beta}{2}}{y}\right)^{\frac{-2n-\alpha-4}{2}} \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)}{y^2} dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{-1} \Gamma\left(\frac{2n+\alpha+2}{2}\right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)}$$

In the same manner the denominator of (6) become as follows

$$\int_0^\infty \frac{1}{\theta^2} \; h(\theta|t) \; d\theta = \frac{\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{-2} \Gamma\left(\frac{2n+\alpha+4}{2}\right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)}$$

Hence,

$$\hat{\theta}_5 = \frac{2\sum_{i=1}^{n} t_i + \beta}{2n+\alpha+2} \qquad \ldots (15)$$

vi) Inverted Chi square prior information, under the general entropy loss function

According to the general entropy loss function, the corresponding Bayes estimator for the parameter θ is derived by substituting the posterior distribution (14) in (7), as follows:

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \int_0^\infty \theta^{\frac{-2n-\alpha-2\gamma-2}{2}} \; e^{\frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)}{\theta}} \, d\theta$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i + \frac{\beta}{2}}{\theta}$$

Then

$$E(\theta^{-\gamma}|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \int_0^\infty \left(\frac{\sum_{i=1}^{n} t_i + \frac{\beta}{2}}{y}\right)^{\frac{-2n-\alpha-2\gamma-2}{2}} e^{-y} \frac{-\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)}{y^2} \, dy$$

And after few steps

$$E[(\theta^{-\gamma}|\boldsymbol{t})]^{-\frac{1}{\gamma}} = \left[\frac{\left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)^{-\gamma} \Gamma\left(\frac{2n+\alpha+2\gamma}{2}\right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)}\right]^{-\frac{1}{\gamma}}$$

Hence,

$$\hat{\theta}_6 = \left(\sum_{i=1}^{n} t_i + \frac{\beta}{2}\right)\left(\frac{\Gamma\left(\frac{2n+\alpha}{2}\right)}{\Gamma\left(\frac{2n+\alpha+2\gamma}{2}\right)}\right)^{\frac{1}{\gamma}} \qquad \dots (16)$$

SIMULATION AND RESULTS

In order to investigate the performance of the estimators obtained in the above section, a simulation study is conducted. Samples of size *n= 25, 50* and *100* are generated from the exponential distribution with two values of θ $(\theta = 0.5, 1)$. Four values for the extension of Jeffreys parameter $(c = 0.5, 1, 1.5, 2)$, and the following pairs of values of the hyper parameters α and β are chosen $\{(\alpha, \beta) = (1, 0.5), (2, 1), (5, 1)\}$. Four values for the GELF parameter γ are chosen as $(\gamma = 1, -1, 2, -2)$. The number of replication used was $R=1000$, and the mean of the estimated values for the parameter θ is obtained along with its mean square error (MSE) to compare the efficiency of the estimators , where

$$\mu(\hat{\theta}) = \frac{\sum_{i=1}^{1000} \hat{\theta}_i}{R}, \qquad \text{and } MSE\,(\hat{\theta}) = \frac{\sum_{i=1}^{1000} \left(\hat{\theta}_i - \theta\right)^2}{R}$$

The results of the simulation study are summarized and tabulated in tables 1-3 for each estimator and for all sample sizes.

Table (1) the mean of Bayes estimators and MSE with the extension of Jeffreys prior under QLF $(\hat{\theta}_1)$ and GELF$(\hat{\theta}_2)$.

$\theta = 0.5, c = 0.5$						
n	Criteria	$(\hat{\theta}_1)$	$(\hat{\theta}_2)$			
			$\gamma = -2$	$\gamma = -1$	$\gamma = 1$	$\gamma = 2$
25	Mean	.4835	.5350	.5237	.5028	.4931
	MSE	.0101	.0133	.0121	.0106	.0103
50	Mean	.4924	.5178	.5124	.5022	.4973
	MSE	.0051	.0059	.0056	.0053	.0052
100	Mean	.4950	.5076	.5050	.5000	.4975
	MSE	.0026	.0028	.0028	.0027	.0027
$\theta = 0.5, c = 1$						
n	Criteria	$(\hat{\theta}_1)$	$(\hat{\theta}_2)$			
			$\gamma = -2$	$\gamma = -1$	$\gamma = 1$	$\gamma = 2$
25	Mean	.4656	.5123	.5028	.4835	.4744
	MSE	.0103	.0112	.0106	.0101	.0101
50	Mean	.4829	.5073	.5022	.4924	.4876
	MSE	.0052	.0054	.0053	.0051	.0051
100	Mean	.4901	.5025	.5000	.4950	.4925
	MSE	.0027	.0027	.0027	.0026	.0027
$\theta = 0.5, c = 2$						
n	Criteria	$(\hat{\theta}_1)$	$(\hat{\theta}_2)$			
			$\gamma = -2$	$\gamma = -1$	$\gamma = 1$	$\gamma = 2$
25	Mean	.4337	.4744	.4656	.4489	.4411
	MSE	.0123	.0101	.0103	.0111	.0117
50	Mean	.4650	.4876	.4829	.4738	.4694
	MSE	.0057	.0051	.0052	.0054	.0055
100	Mean	.4807	.4926	.4902	.4854	.4831
	MSE	.0028	.0027	.0027	.0027	.0028
$\theta = 1, c = 0.5$						
n	Criteria	$(\hat{\theta}_1)$	$(\hat{\theta}_2)$			
			$\gamma = -2$	$\gamma = -1$	$\gamma = 1$	$\gamma = 2$
25	Mean	.9670	1.0701	1.0475	1.0056	.9861
	MSE	.0404	.0531	.0484	.0426	.0411
50	Mean	.9848	1.0356	1.0250	1.0045	.9946
	MSE	.0205	.0237	.0226	.0211	.0207
100	Mean	.9900	1.0152	1.0100	.9999	.9950
	MSE	.0106	.0113	.0110	.0107	.0106
$\theta = 1, c = 1.5$						
n	Criteria	$(\hat{\theta}_1)$	$(\hat{\theta}_2)$			
			$\gamma = -2$	$\gamma = -1$	$\gamma = 1$	$\gamma = 2$
25	Mean	.8979	.9861	.9670	.9311	.9144
	MSE	.0444	.0411	.0404	.0412	.0425
50	Mean	.9476	.9946	.9848	.9658	.9567
	MSE	.0215	.0207	.0205	.0207	.0210
100	Mean	.9708	.9950	.9900	.9803	.9756
	MSE	.0109	.0106	.0106	.0107	.0108

Table (2) the mean of Bayes estimators and MSE with inverted gamma prior under QLF $(\hat{\theta}_3)$ and GELF$(\hat{\theta}_4)$.

θ = 0.5, α = 1, β = 0.5						
n	Criteria	$(\hat{\theta}_3)$	$(\hat{\theta}_4)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.4841	.4933	.5027	.4838	.5228
	MSE	.0094	.0095	.0098	.0097	.0111
50	Mean	.4925	.4973	.5022	.4924	.5122
	MSE	.0049	.0050	.0051	.0050	.0054
100	Mean	.4951	.4975	.5000	.4950	.5050
	MSE	0.0026	.0026	.0026	.0026	.0027
θ = 0.5, α = 2, β = 1						
n	Criteria	$(\hat{\theta}_3)$	$(\hat{\theta}_4)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.4846	.4936	.5026	.4841	.5219
	MSE	.0087	.0088	.0091	.0094	.0103
50	Mean	.4927	.4974	.5021	.4925	.5120
	MSE	.0047	.0048	.0049	.0049	.0052
100	Mean	.4951	.4975	.5000	.4951	.5049
	MSE	.0025	.0026	.0026	.0026	.0026
θ = 0.5, α = 5, β = 1						
n	Criteria	$(\hat{\theta}_3)$	$(\hat{\theta}_4)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.4378	.4450	.4523	.4586	.4679
	MSE	.0108	.0101	.097	.0099	.0089
50	Mean	.4662	.4705	.4747	.4787	.4835
	MSE	.0053	.0051	.0050	.0051	.0048
100	Mean	.4811	.4834	.4857	.4879	.4904
	MSE	.0027	.0027	.0026	.0026	.0026
θ = 1, α = 2, β = 1						
n	Criteria	$(\hat{\theta}_3)$	$(\hat{\theta}_4)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.9336	.9507	.9682	.9497	1.0054
	MSE	.0383	.376	.0375	.0390	.0394
50	Mean	.9665	.9757	.9851	.9754	1.0044
	MSE	.0199	.0197	.0197	.0201	.0203
100	Mean	.9805	.9853	.9901	.9852	.9999
	MSE	.0105	.0106	.0105	.0104	.0104
θ =1, α = 5, β = 1						
n	Criteria	$(\hat{\theta}_3)$	$(\hat{\theta}_4)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.8432	.8572	.8714	.8997	.9014
	MSE	.0523	.0490	.0461	.0428	.0414
50	Mean	.9147	.9230	.9313	.9481	.9486
	MSE	.0241	.0230	.0221	.0211	.0207
100	Mean	.9528	.9573	.9618	.9710	.9711
	MSE	.0118	.0114	.0112	.0108	.0107

Table (3) The mean of Bayes estimators and MSE with inverted Chi square prior under QLF $(\hat{\theta}_5)$ and GELF$(\hat{\theta}_6)$.

θ = 0.5, α = 1, β = 0.5						
n	Criteria	$(\hat{\theta}_5)$	$(\hat{\theta}_6)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.5336	.5343	.5233	.5028	.4932
	MSE	.0122	.0127	.0116	.0102	.0099
50	Mean	.5174	.5176	.5123	.5022	.4973
	MSE	.0057	.0058	.0055	.0052	.0051
100	Mean	.5075	.5075	.5050	.5000	.4975
	MSE	.0028	.0028	.0027	.0027	.0026
θ = 0.5, α = 2, β = 1						
n	Criteria	$(\hat{\theta}_5)$	$(\hat{\theta}_6)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.5323	.5336	.5228	.5027	.4933
	MSE	.0113	.0122	.0112	.0098	.0095
50	Mean	.5171	.5174	.5122	.5022	.4973
	MSE	.0055	.0057	.0054	.0051	.0050
100	Mean	.5074	.5075	.5050	.5000	.4975
	MSE	.0027	.0028	.0027	.0026	.0026
θ = 0.5, α = 5, β = 1						
n	Criteria	$(\hat{\theta}_5)$	$(\hat{\theta}_6)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.4762	.5028	.4932	.4753	.4669
	MSE	.0088	.0099	.0095	.0094	.0096
50	Mean	.4881	.5022	.4973	.4878	.4832
	MSE	.0047	.0051	.0050	.0049	.0050
100	Mean	.4927	.5000	.4975	.4927	.4903
	MSE	.0025	.0026	.0026	.0026	.0026
θ = 1, α = 2, β = 1						
n	Criteria	$(\hat{\theta}_5)$	$(\hat{\theta}_6)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	1.0253	1.0468	1.0256	.9862	.9678
	MSE	.0416	.0465	.0432	.0395	.0389
50	Mean	1.0144	1.0248	1.0145	.9946	.9850
	MSE	.0209	.0221	.0213	.0203	.0201
100	Mean	1.0049	1.0100	1.0049	.9950	.9901
	MSE	.0106	.0109	.0107	.0105	.0105
θ = 1, α = 5, β = 1						
n	Criteria	$(\hat{\theta}_5)$	$(\hat{\theta}_6)$			
			γ = -2	γ = -1	γ = 1	γ = 2
25	Mean	.9174	.9864	.9676	.9324	.9159
	MSE	.0396	.0396	.0389	.0398	.0410
50	Mean	.9575	.9946	.9849	.9661	.9571
	MSE	.0202	.0203	.0201	.0203	.0206
100	Mean	.9758	.9950	.9901	.9804	.9757
	MSE	.0106	.0105	.0105	.0106	.0107

DISCUSSION

The results of this study are stated in the following points:

The effect of prior information parameters

It is noted that Bayes estimators based on the extension of Jeffreys prior information are generally underestimated with the increase of c. The extent of underestimation is higher for small n. On the other hand the estimates of the parameter based on both the inverted gamma and the inverted chi square priors are observed to be underestimated with the increase of θ and the hyper parameter α. The extent of underestimation is higher in the case of small n. In general Bayes estimator based on the inverted gamma prior is the best when the hyper parameters α and β are close.

The effect of loss function

Comparison with respect to loss functions shows that Bayes estimators of the parameter θ under GELF are better than QLF for certain values of γ and depending on the values of the prior parameters. From table 1 we can easily observe that results are better with positive γ and small c. Positive γ is also better in tables *2* and *3* except the cases when $\alpha = 5$ and $\beta = 1$, were estimators with negative γ are better. Under both the QLF and GELF with negative γ we can observe that the parameters are overestimated with the inverted chi square prior with small α and β, and the extent of overestimation is higher when n is small.

The effect of sample size

It is apparent from Tables (1-3) that MSE of all the estimators decreases notably as the sample size increases.

Finally; and after an extensive study of the results, we suggest the use of the GELF with proper choice of γ, as a suitable alternative of the QLF when the loss is asymmetric in nature. The comparison of the informative and non-informative priors shows that informative prior performs better than non-informative prior. Accordingly; if adequate information is available about the parameters it is preferable to use conjugate informative priors, otherwise the extension of Jeffreys' prior gives quite reasonable results.

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