Some Probability Characteristics of the Solution of Stochastic Fredholm Integral Equation Contains a Joint Gamma Process

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ABSTRACT
In this paper, some probability characteristics functions (probability density and spectral density) are derived depending upon the smallest variance of the stochastic solution of supposing stochastic linear Fredholm integral equation of the second kind containing a joint Gamma process found by Adomian decomposition method (A.D.M).

Key Words: A.D.M., Fredholm Integral Equation, Joint Gamma Function Process.

بعض المزايا الاحتمالية لحل معادلة فريدهولم التكاملية والعشوائية التي تحتوي على عملية كاما المرتبطة

الخلاصة
في هذا البحث بعض دوال المزايا الاحتمالية (كثافة الاحتمالية، والكثافة الطيفية) تم اشتقاقها استناداً لأصغر تباين لحل معادلة افتراضية لفريدهوم التكاملية من النوع الثاني التي تحوي على عملية گاما المرتبطة و العشوائية والذي تم ايجاده بطريقة ادوميان التحليلية.

INTRODUCTION
In the beginning of 1980's, a new method for solving linear and non-linear integral (differential) equations for various kinds has been proposed by G. Adomian, the so called Adomian decomposition method, [1]. After that, in recent years many researchers had been used this method to solve analytically (numerically) a stochastic Fredholm integral equations or especial kinds of linear (non-linear) Fredholm integral equations, [2, 3, 4]. Most of those researchers are just interested as a final goal in finding the numerical solution on some definite closed interval to study a unique comparison between the numerical solution of a given Fredholm integral equation and its exact solution.

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In this paper our goal is not only interesting in the solution of the supposing stochastic Fredholm integral equation but we concentrate ourselves in the derivation of many probability characteristics of this solution (mean, variance, characteristic function, correlation function and spectral density function) that is by depending upon the smallest value of the variance of this solution as a basis for that.

PRELIMINARIES

The gamma process $\Gamma(w,\alpha,\lambda,t)$ is a Levy process whose marginal distribution at time $t > 0$ is a gamma distribution with mean $\frac{\alpha t}{\lambda}$ and variance $\frac{\alpha t}{\lambda^2}$ (i.e.)

$$\Gamma(w,\alpha,\lambda,t)=\frac{(\lambda)^{\alpha t}}{\Gamma(\alpha t)}w^{\alpha t-1}e^{-\lambda w},\ w > 0,\ t > 0 \qquad \ldots(1)$$

Where the parameter α controls the rate of jump arrivals (shape parameter) and the scaling parameter λ inversely controls the jump size (scale parameter).
Moreover, the gamma process has the following properties, [4]

(i) $\Gamma(w,\alpha,\lambda,t) = 0$

(ii) For any $0\le t_1 < t_2 <\ldots< t_n <\infty$; $\Gamma(w,\alpha,\lambda,t_2) - \Gamma(w,\alpha,\lambda,t_1)$, …, $\Gamma(w,\alpha,\lambda,t_n) - \Gamma(w,\alpha,\lambda,t_{n-1})$ are independent increments.

(iii) For any $s < t$; $\Gamma(w,\alpha,\lambda,t) - \Gamma(w,\alpha,\lambda,s)$.

Now, we consider the following two dimensional stochastic Fredholm integral equations,

$$u(w_1,w_2,t)=\Gamma(w_1,w_2,\alpha,\lambda,t)+\int_0^\infty\int_0^\infty k(w_1,w_2,s_1,s_2,t)U(s_1,s_2,t)ds_1ds_2 \qquad \ldots(2)$$

Where

- $\Gamma(w_1,w_2,\alpha,\lambda,t)=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(w_1w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}$ is a joint Gamma process $\qquad \ldots(3)$
- $k(w_1,w_2,s_1,s_2,t)$ are known stochastic kernel defined by $t > 0$ and $s_1, s_2 \in S$, S is a compact metric space and having respectively the supposing formulas

$$k(w_1,w_2,s_1,s_2,t)=e^{-[(s_1+s_2)+(w_1+w_2)]t} \qquad \ldots(4)$$

- $u(s_1,s_2,t)$ are scalar functions defined for $t > 0$, $s_1, s_2 > 0$.

By substituting equations (3) and (4) into equation (2), we get

$$u(w_1,w_2,t)=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(w_1w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}+\int_0^\infty\int_0^\infty A\,e^{-[(s_1+s_2)+(w_1+w_2)]t}u(s_1,s_2,t)ds_1ds_2 \qquad \ldots(5)$$

Remark (2.1)

In this paper three cases for the parameters (α,λ) will be considered,

1) $\alpha < \lambda$; $\alpha = 0.5$, $\lambda = 2$.

2) $\alpha = \lambda$; $\alpha = \lambda = 1$.
3) $\alpha > \lambda$; $\alpha = 2$, $\lambda = 0.5$.

Now, to find the solution of the equation (5), Adomain decomposition method will be used which briefly depend on the following steps, [5],

$$u_0(w_1,w_2,t) = \Gamma(w_1,w_2,\alpha,\lambda,t) = \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(w_1 w_2)^{\alpha t-1} e^{-\lambda(w_1+w_2)} \qquad \ldots(6)$$

$$u_0(s_1,s_2,t) = \Gamma(s_1,s_2,\alpha,\lambda,t) = \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(s_1 s_2)^{\alpha t-1} e^{-\lambda(s_1+s_2)} \qquad \ldots(7)$$

and

$$u_{m+1}(w_1,w_2,t) = \int_0^\infty\int_0^\infty e^{-[(s_1+s_2)+(w_1+w_2)]t}\, u_m(s_1,s_2,t)\, ds_1 ds_2,\ m=0,1,2,\ldots \qquad \ldots(8)$$

That is to get the solution

$$u(w_1,w_2,t) = u_0(w_1,w_2,t) + \sum_{n=1}^{\infty} u_n(w_1,w_2,t) \qquad \ldots(9)$$

So, for the first iteration (m = 0), by equations (7) and (8),

$$u_1(w_1,w_2,t) = \int_0^\infty\int_0^\infty e^{-[(s_1+s_2)+(w_1+w_2)]t}\, \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(s_1 s_2)^{\alpha t-1} e^{-\lambda(s_1+s_2)} ds_1 ds_2$$

$$= \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left\{\left[\int_0^\infty e^{-(s_1+w_1)t} s_1^{\alpha t-1} e^{-\lambda s_1} ds_1\right]\cdot\left[\int_0^\infty e^{-(s_2+w_2)t} s_2^{\alpha t-1} e^{-\lambda s_2} ds_2\right]\right\}$$

$$= \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left\{\left[e^{-w_1 t}\int_0^\infty s_1^{\alpha t-1} e^{-\frac{s_1}{1/t+\lambda}} ds_1\right]\cdot\left[e^{-w_2 t}\int_0^\infty s_2^{\alpha t-1} e^{-\frac{s_2}{1/t+\lambda}} ds_2\right]\right\}$$

$$= \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left\{\left[e^{-w_1 t}\frac{\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}}{\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}}\int_0^\infty s_1^{\alpha t-1} e^{-\frac{s_1}{1/t+\lambda}} ds_1\right]\cdot\right.$$

$$\left.\left[e^{-w_2 t}\frac{\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}}{\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}}\int_0^\infty s_2^{\alpha t-1} e^{-\frac{s_2}{1/t+\lambda}} ds_2\right]\right\}$$

or

$$= \frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left\{\left[e^{-w_1 t}\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}\right]\cdot\right.$$

$$\left.\left[e^{-w_2 t}\Gamma(\alpha t)\left(\frac{1}{t+\lambda}\right)^{\alpha t}\int_0^\infty s_2^{\alpha t-1} e^{-\frac{s_2}{1/t+\lambda}} ds_2\right]\right\}$$

$$u_1(w_1,w_2,t) = \left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t} e^{-(w_1+w_2)t} \qquad \ldots(10)$$

and for the second iteration (m = 1), by equations (8) and (10),

$$u_2(w_1,w_2,t)=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\int_0^\infty\int_0^\infty e^{-[(s_1+s_2)+(w_1+w_2)]t}\,e^{-(s_1+s_2)t}\,ds_1ds_2$$

$$=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}e^{-(w_1+w_2)t}\int_0^\infty\int_0^\infty e^{-2(s_1+s_2)t}\,ds_1ds_2$$

$$=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}e^{-(w_1+w_2)t}\left\{\left[\int_0^\infty e^{-2s_1t}ds_1\right]\left[\int_0^\infty e^{-2s_2t}ds_2\right]\right\}$$

$$=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}e^{-(w_1+w_2)t}\left(-\frac{1}{2t}e^{-2s_1t}\Big|_0^\infty\right)\left(-\frac{1}{2t}e^{-2s_2t}\Big|_0^\infty\right)$$

$$=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}e^{-(w_1+w_2)t}\left(\frac{1}{2t}\right)\left(\frac{1}{2t}\right)$$

or

$$u_2(w_1,w_2,t)=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\frac{e^{-(w_1+w_2)t}}{4t^2} \qquad \ldots(11)$$

and by repeating iteration for m = 2, 3, … and adding (10), (11), we get,

$$\sum_{n=1}^{\infty}u_n(w_1,w_2,t)=\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}e^{-(w_1+w_2)t}\sum_{r=0}^{\infty}\left(\frac{1}{4t^2}\right)^r$$

where $\sum_{r=0}^{\infty}\left(\frac{1}{4t^2}\right)^r$ is a geometrical series for $t \ge 0.5$.

$$\sum_{n=1}^{\infty}u_n(w_1,w_2,t)=4\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\frac{t}{4t-1}e^{-(w_1+w_2)t} \qquad \ldots(12)$$

Finally, by substituting (6), (12) into (9) which represents the solution of (5), we get,

$$u(w_1,w_2,t)=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(w_1w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}+ 4\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)e^{-(w_1+w_2)t}, \qquad \ldots(13)$$

where $w_1, w_2 > 0$, $t \ge 0.5$, $\alpha > 0$, $\lambda > 0$

Moreover, the solution (13) can be also considered as a solution over the interval $0 < w_1 \le 1$, $0 < w_2 \le 1$.

STATISTICAL PROPERTIES OF THE STOCHASTIC SOLUTIONS

In order to derive the statistical properties of the solution (13) over the interval $0 < w_1, w_2 \le 1$, $t \ge 0.5$, it must be that each of them is a joint probability density function (j.p.d.f.). So, we multiply the second term of (13) by A and equate the integral by one that is to find A which make each stochastic solution is a j. p.d.f. of (w_1,w_2,t), (i.e.), we write,

$$\int_0^1\int_0^1\left[\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}(w_1 w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}+4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)e^{-(w_1+w_2)t}\right]dw_1dw_2=1$$

Or

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\int_0^1\int_0^1(w_1 w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}dw_1dw_2+4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\int_0^1\int_0^1 e^{-(w_1+w_2)t}dw_1dw_2=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\int_0^1\int_0^1 w_1^{\alpha t-1}e^{-\lambda w_1}w_2^{\alpha t-1}e^{-\lambda w_2}dw_1dw_2+4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\int_0^1\int_0^1 e^{-(w_1+w_2)t}dw_1dw_2=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\int_0^1\int_0^1 w_1^{\alpha t-1}(1-\lambda w_1+\frac{(\lambda w_1)^2}{2!}-\ldots)w_2^{\alpha t-1}(1-\lambda w_2+\frac{(\lambda w_2)^2}{2!}-\ldots)dw_1dw_2+$$

$$4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\int_0^1\int_0^1 e^{-(w_1+w_2)t}dw_1dw_2=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\int_0^1\int_0^1 w_1^{\alpha t-1}-\lambda w_1^{\alpha t}+\frac{\lambda^2 w_1^{\alpha t+1}}{2!}-\ldots)(w_2^{\alpha t-1}-\lambda w_2^{\alpha t}+\frac{\lambda^2 w_2^{\alpha t+1}}{2!}-\ldots)dw_1dw_2+$$

$$4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\int_0^1\int_0^1 e^{-(w_1+w_2)t}dw_1dw_2=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\int_0^1\int_0^1\left(\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_1^{\alpha t+n-1}}{n!}\right)\left(\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_2^{\alpha t+n-1}}{n!}\right)dw_1dw_2+$$

$$4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\int_0^1\int_0^1 e^{-(w_1+w_2)t}dw_1dw_2=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_1^{\alpha t+n-1}}{n!}dw_1\right)\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{\lambda^2 w_2^{\alpha t+n-1}}{n!}dw_2\right)+$$

$$4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\left(\int_0^1 e^{-w_1 t}dw_1\right)\left(\int_0^1 e^{-w_2 t}dw_2\right)=1$$

$$\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\left(\frac{e^{-t}-1}{t}\right)^2=1$$

$$4A\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\left(\frac{e^{-t}-1}{t}\right)^2=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1$$

Hence,

$$A=\frac{\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{4\left(\frac{\lambda}{t+\lambda}\right)^{2\alpha t}\left(\frac{t^2}{4t^2-1}\right)\left(\frac{e^{-t}-1}{t}\right)^2},\ \alpha,\lambda,t>0.5 \qquad \ldots(14)$$

The solution of equation (13) is j.p.d. function when,

$$u(w_1,w_2,t)=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[(w_1w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}\right]+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}e^{-(w_1+w_2)t} \qquad \ldots(15)$$

$,\alpha,\lambda,t>0.5$

First and Second Mean

First mean:

$$E_u\left[(w_1,w_2,t)\right]=\int_0^1\int_0^1(w_1w_2)\left[u(w_1,w_2,t)\right]dw_1dw_2$$

We start by the first mean of $u(w_1,w_2,t)$,

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\int_0^1\int_0^1(w_1,w_2)(w_1,w_2)^{\alpha t-1}e^{-\lambda(w_1+w_2)}dw_1dw_2+\int_0^1\int_0^1(w_1,w_2)\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}e^{-(w_1+w_2)t}dw_1dw_2\right]$$

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}[\ ]\left[\int_0^1\int_0^1 w_1^{\alpha t}e^{-\lambda w_1}w_2^{\alpha t}e^{-\lambda w_2}dw_1dw_2+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\int_0^1\int_0^1(w_1,w_2)e^{-(w_1+w_2)t}dw_1dw_2\right]$$

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\int_0^1\int_0^1 w_1^{\alpha t}(1-\lambda w_1+\frac{(\lambda w_1)^2}{2!}-...)w_2^{\alpha t}(1-\lambda w_2+\frac{(\lambda w_2)^2}{2!}-...)dw_1dw_2+\right.$$
$$\left.\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\int_0^1\int_0^1 w_1e^{-w_1t}w_2e^{-w_2t}dw_1dw_2\right]$$

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_1^{\alpha t+n}}{n!}dw_1\right)+\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_2^{\alpha t+n}}{n!}dw_2\right)+\right.$$
$$\left.\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\left(\int_0^1 w_1e^{-w_1t}dw_1\right)\left(\int_0^1 w_2e^{-w_2t}dw_2\right)\right]$$
$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n+1)}\right)^2+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\left[\frac{1}{t^2}(1-(1+t)e^{-t})\right]^2\right]$$

or

$$E_u\left[(w_1,w_2,t)\right]=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n+1)}\right)^2+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{(e^{-t}-1)^2}\left[\frac{1-(1+t)e^{-t}}{t}\right]^2\right] \quad \ldots(16)$$

$\alpha, \lambda, t > 0.5,\ 0 < w_1,\ w_2 \le 1$

Second mean:

$$E_u\left[(w_1,w_2,t)\right]=\int_0^1\int_0^1 (w_1w_2)^2u(w_1,w_2,t)\,dw_1dw_2$$

It is easy to derive the second mean of $u(w_1,w_2,t)$, it will be,

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_1^{\alpha t+n+1}}{n!}dw_1\right)+\left(\int_0^1\sum_{n=0}^{\infty}(-\lambda)^n\frac{w_2^{\alpha t+n+1}}{n!}dw_2\right)+\right.$$

$$\left.\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\left(\int_0^1 w_1^2e^{-w_1t}dw_1\right)\left(\int_0^1 w_2^2e^{-w_2t}dw_2\right)\right]$$

$$=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n+1)}\right)^2+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)^2}\left[\frac{2-(t^2+2t+2)e^{-t}}{t^3}\right]^2\right]$$

$$E_u\left[(w_1,w_2,t)\right]=\frac{(\lambda)^{2\alpha t}}{[\Gamma(\alpha t)]^2}\left[\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n+2)}\right)^2+\frac{\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!(\alpha t+n)}\right)^2-1}{\left(\frac{e^{-t}-1}{t}\right)}\left[\frac{2-(t^2+2t+2)e^{-t}}{t^3}\right]^2\right] \quad \ldots(17)$$

$\alpha, \lambda, t > 0.5$, $0 < w_1$, $w_2 \le 1$

Calculations of the variance of the probability density function (p.d.f.) (15) by the formula $var[u(w_1,w_2,t)] = E_u[(w_1^2, w_2^2,t)] - \{E_u[(w_1,w_2,t)]\}^2$ for three cases (remark (2.1)) when $t \ge 0.55$ are tabulated in Table (1) of the appending attached to this paper. It is noted that, these variances in the three cases are slowly decreasing to tends to zero when t is relatively large.

The minimum variances from Table (1) is presented in the first, second, and third rows in Table (3.1), respectively.

Table (3.1) presents the minimum variances from Table (1).

p.d.f.	**t**	**λ = α = 1**	**λ > α; λ = 2, α = 0.5**	**λ < α; λ = 0.5, α = 2**
$u(w_1,w_2,t)$	0.55	0.026769	0.037453	0.001824
	0.85	0.000841	-------	-------
	1.20	-------	0.001161	-------

Furthermore, the probability density function (15) with respect to the three cases remark (2.1) can be rewritten respectively as follows

(1) $\alpha < \lambda$; $\alpha = 0.5$, $\lambda = 2$

$$u(w_1, w_2, t) = 0.240465(w_1w_2)^{\frac{1}{10}} \exp(-0.5w_1 - 0.5w_2) - 0.203448\exp\left[-(w_1 + w_2)^{\frac{11}{20}}\right]$$

(2) $\alpha = \lambda$; $\alpha = \lambda = 1$

$$u(w_1, w_2, t) = 3\times10^{-20}\left[\frac{269334\exp(-w_1 - w_2) - 2366533\exp\left[-(w_1w_2)^{\frac{17}{20}}(w_1w_2)^{\frac{3}{20}}\right]}{(w_1w_2)^{\frac{3}{20}}}\right]$$

(3) $\alpha > \lambda$; $\alpha = 2$, $\lambda = 0.5$

$$u(w_1, w_2, t) = 5.00000\times10^{-20}\left[\frac{207187\exp(-2w_1 - 2w_2) - 898158\exp\left[-(w_1 + w_2)^{\frac{6}{5}}(w_1w_2)^{\frac{2}{5}}\right]}{(w_1w_2)^{\frac{2}{5}}}\right]$$

Autocovariance Function

For any $s > t$, $s - t = \tau$, the autocovariance function $B(\tau)$ of the independent p.d.f. functions $u(w_1,w_2,t)$ and $u(w_1,w_2, t + \tau)$ is an even function depends on the difference $|\tau| = |s - t| = |t - s|$ and can be found as follows, [6].

$$B(\tau) = E_u((w_1,w_2,t)\,(w_1,w_2,t + \tau)) - E_u(w_1,w_2,t)\, E_u(w_1,w_2, t + \tau)$$
$$= E_u((w_1^2, w_2^2,t) + E_u(w_1,w_2,t)(w_1,w_2,t + \tau) - E_u(w_1^2, w_2^2,t) - E_u(w_1,w_2,t)\, E_u(w_1,w_2,t + \tau))$$
$$= E_u((w_1^2, w_2^2,t) + E_u(w_1,w_2,t)\, E_u(w_1,w_2,t + \tau) - E_u(w_1^2, w_2^2,t) - E_u(w_1,w_2,t)\, E_u(w_1,w_2,t+\tau))$$
$$= E_u((w_1^2, w_2^2,t) + E_u(w_1,w_2,t)\,(E_u(w_1,w_2,t+\tau) - E_u(w_1^2, w_2^2,t)) - E_u(w_1,w_2,t)\, E_u(w_1,w_2,t+\tau)$$
$$= E_u(w_1^2, w_2^2,t) + E_u(w_1,w_2,t)E_u(w_1,w_2,t+\tau) - (E_u(w_1,w_2,t))^2 - E_u(w_1,w_2,t)\, E_u(w_1,w_2,t+\tau)$$

or

$$B(\tau)=E_u(w_1^2, w_2^2,t) - (E_u(w_1,w_2,t))^2 = \text{var } u(w_1,w_2,t), \quad \tau > 0,\ 0 \le w_1, w_2 \le 1 \qquad \ldots(18)$$

So, by table (3.1) and equation (18) either when $u(w_1,w_2,t)$, $t = 0.85$, $t = 0.55$ and $t = 1.20$ the autocovariance functions for them with respect to the three cases (remark (2.1)) .

Table (3.2) presents the Autovariance functions of $u(w_1,w_2,t)$ and $u(w_1,w_2,t+\tau)$.

t	$\lambda = \alpha = 1$	$\lambda > \alpha$; $\lambda = 2$, $\alpha = 0.5$	$\lambda < \alpha$; $\lambda = 0.5$, $\alpha = 2$
0.85	0.026769	0.037453	0.001824
0.55	0.000841	-------	-------
1.20	-------	0.001161	-------

Spectral Density Function

The spectral density functions (s.d.f) of $u(w_1,w_2,t)$ is an even function represents the average power in the p.d.f $u(w_1,w_2,t)$ at the angular frequency $0 < \theta < 2n\pi$, $n \in I^+$ and can be found by Khinchin's formula as follows, [7]

$$f(u,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\tau)e^{i\theta}d\tau$$

$$= \frac{B(\tau)}{2\pi}\int_{-\infty}^{\infty}(\cos\theta\tau - i\sin\theta\tau)\,d\tau$$

and for $\tau = s - t > 0$

$$f(u,\theta) = \frac{B(\tau)}{\pi}\int_{0}^{s-t}(\cos\theta\tau - i\sin\theta\tau)\,d\tau$$

or

$$f(u,\theta) = \frac{B(\tau)}{\pi}\cdot\frac{\sin[(s-t)\theta]}{\theta}, 0 < \theta \le 2n\pi,\ n \in I^+ \qquad \ldots(19)$$

So, by Table (3.2) and Equation (19) either when u (w_1, w2, t), t = 0.85, t = 0.55 and t = 1.20 the s.d.f of them with respect to the three cases (remark (2.1)) can respectively be written as follows:

(1) $\lambda = \alpha = 1$

$$f(u,\theta) = \frac{0.000841}{\pi}\cdot\frac{\sin[(s-0.85)\theta]}{\theta}, 0 < \theta \le 2\pi \qquad \ldots(20a)$$

(2) $\lambda < \alpha$, $\lambda = 0.5$, $\alpha = 2$

$$f(u,\theta) = \frac{0.001824}{\pi}\cdot\frac{\sin[(s-0.55)\theta]}{\theta}, 0 < \theta \le 2\pi \qquad \ldots(20b)$$

(3) $\lambda > \alpha$, $\lambda = 2$, $\alpha = 0.5$

$$f(u,\theta) = \frac{0.001161}{\pi}\cdot\frac{\sin[(s-1.20)\theta]}{\theta}, 0 < \theta \le 2\pi \qquad \ldots(20c)$$

Figures (4, 5 and 6) represent the graphs of three pairs (f (u,θ)) corresponding to the three cases (remark (2.1)) and s = 2, 9 just be chosen to complete the figures.

DISCUSSION AND RECOMMENDATION

With respect to the three considering cases of the parameters of the joint Gamma process (α,λ), the variances of the resulting p.d.f are slowly decreasing when the time t is increasing up to reach some values.

Furthermore, the smallest variance is when the case of the two parameters (α,λ) are equal to one at t = 0.85.

For the future work, we recommend to consider more different cases of the parameters of the Gamma process with different values that is no derive the largest variance of the resulting p.d.f.

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Table (1) Variance of $u(w_1,w_2,t)$.

t	$\lambda < \alpha$; $\lambda = 0.5, \alpha = 2$	$\lambda = \alpha$; $\lambda = \alpha = 1$	$\lambda > \alpha$; $\lambda = 2, \alpha = 0.5$
0.05	0.047567	0.047919	0.047909
0.10	0.44418	0.046756	0.047133
0.15	0.0403681	0.045278	0.046289
0.20	0.035715	0.043588	0.045385
0.25	0.030526	0.041743	0.044425
0.30	0.024945	0.039760	0.043412
0.35	0.019259	0.037627	0.042346
0.50	0.013826	0.035308	0.041224
0.45	0.008979	0.032758	0.040040
0.50	0.004946	0.029926	0.038787
0.55	**0.001824**	0.026769	0.037453
0.60	no variance	0.023258	0.036023
0.65		0.019383	0.034477
0.70		0.015157	0.032794
0.75		0.010615	0.030948
0.80		0.005817	0.028908
0.85		**0.000841**	0.026643
0.90		no variance	0.024116
0.95		no variance	0.021290
1.00		no variance	0.018121
1.05			0.014569
1.10			0.010588
1.15			0.006134
1.20			**0.001161**
1.25			no variance
1.30			no variance
1.35			no variance
1.40			no variance
1.45			no variance
1.50			no variance

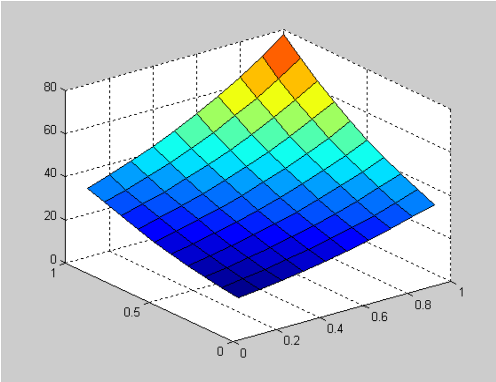

Figure (1)
The graph of $u_1(w_1,w_2,0.85)$, $0 \le w_1,\ w_2 \le 1$ when $S = 2$, 9when $\lambda = \alpha = 1$

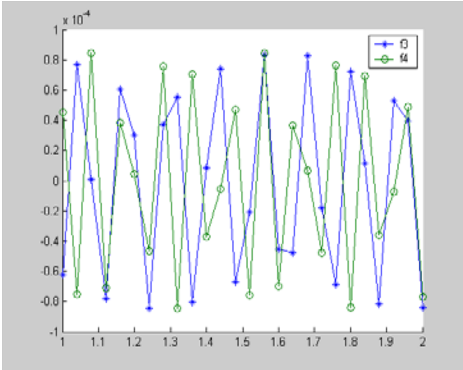

Figure (4)
The graph of $f(u_3,\theta)$ for $0 < \theta < 2\pi$ when $\lambda = \alpha = 1$

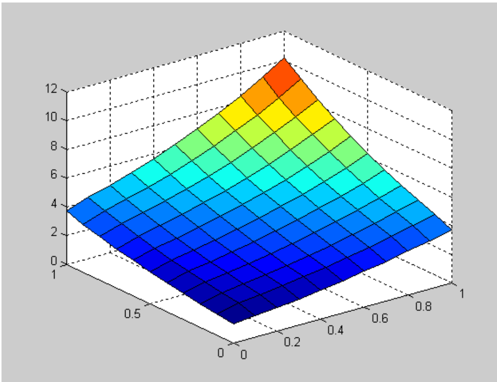

Figure (2)
The graph of $u_2(w_1,w_2,0.55)$, $0 \le w_1,\ w_2 \le 1$ when $\alpha > \lambda$, $\alpha = 2$, $\lambda = 0.5$

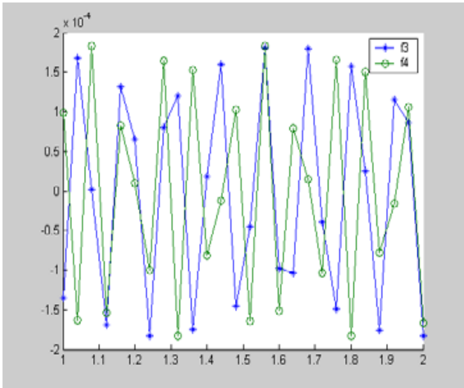

Figure (5)
The graph of $f(u_3,\theta)$ for $0 < \theta < 2\pi$ when $S = 2$, 9 when $\alpha > \lambda$, $\alpha = 2$, $\lambda = 0.5$

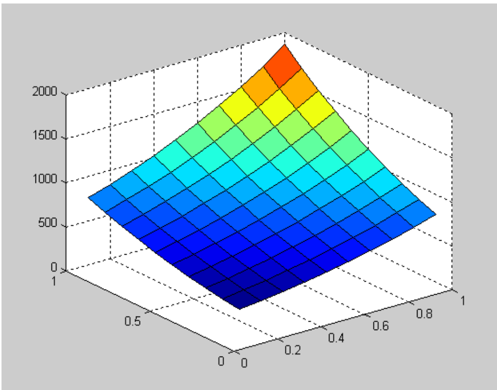

Figure (3)
The graph of $u_2(w_1,w_2,1.20)$, $0 \le w_1,\ w_2 \le 1$ 2π when $S =2$, 9when $\alpha < \lambda$, $\alpha = 0.5$, $\lambda = 2$

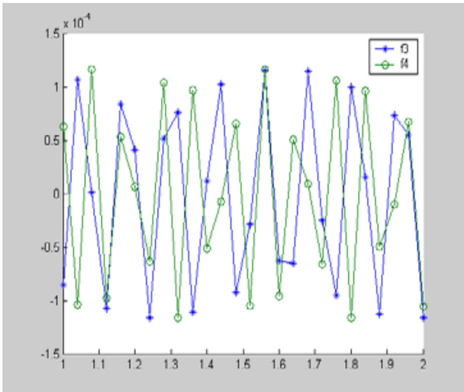

Figure (6)
The graph of $f(u_3,\theta)$ for $0 < \theta <$ when $\alpha < \lambda$, $\alpha = 0.5$, $\lambda = 2$