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Existence of Some New Classes of Semilinear Unbounded Perturbed Operator Equations

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ABSTRACT

In this paper, the adapted approach of existence based on Leray-Schauder degree theorem. The necessary theorems of nonexpansive perturbed operators, lemmas and propositions for existence and uniqueness of proposed classes of semilinear perturbed unbounded operator equations have been adapted and developed with proofs and supported by some illustrative examples.

وجدانية بعض الاصناف الجديده لمعادلات ذات المؤثرات الشبه خطيه المقلقله غير المقيده

المستخلص

في هذا البحث تم اعداد اسلوب لوجدانية الحل بنية على نظريه درجة ليري شويدر.النظريات الضروريه للمؤثرات المقلقله الغير موسعه وبعض القضايا المساعده للوجدانيه والوحدانيه للصنوف المقترحه من المعادلات ذات المؤثرات الشبه خطيه المقلقله غير المقيده تم اعدادها ببراهين متطوره وكذلك تم اسنادها ببعض الامثلة التوضيحيه.

INTRODUCTION

Let H be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$.

In [13], general class of nonlinear operators $F(x)=\mu x\ ,\mu>0$ in a Banach space, which we call projectionally-compact (*P*-compact) and which, among others, contains completely continuous, quasi-compact, and monotone operators and general fixed point theorem for deduce in a simple way the fixed point theorems of Schauder, Rothe, Kaniel and others,have been established.

To study the collection of nonlinear operator $F(.)$ and linear operator A to get the semilinear operator equation, the following literatures are very useful.

Locker in [7], the semilinear equation $Ax=F(x)$ was considered, where *A* is linear, $N(A)$ denotes the null space of *A* and *A** denotes the adjoint of *A* in case

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$D(A)$ is dense in Hilbert space *H*. Let *A* be a closed linear operator in *H* with the following properties:

1) $D(A)$ is dense in *H*, $R(A)$ is closed in *H*,

2) $\dim N(A) = p < \infty$ and $\dim N(A^*) = q < \infty$, $D(A) \bigcap D(F) \neq \emptyset$.

Under these assumptions, the equation $Ax = F(x)$ has at least one solution.

Switzerland [15] has studied the solvability of semilinear operator equations of the form $Au = F(u)$, where $A : D(A) \subset H \to H$ is a self-adjoint linear operator in the real Hilbert space H, and *F* is an appropriately defined nonlinear potential operator. Equations of this type can be considered as abstract formulations for a variety of problems in differential equations as examples:

1. The solutions *x* of the elliptic boundary value problem

$$-\Delta x = f(t, x) \text{ in } \Omega,\ x = 0 \text{ on } \partial\Omega,$$

Where

Ω is a bounded smooth domain in $\mathbb{R}^n$.

2. The existence of $2\pi - periodic$ solutions of the semilinear wave equation

$$u_{tt} - u_{xx} = f(x, t, u) \text{ in } (0, \pi) \times \mathbb{R},$$

$$u(0,t) = u(\pi, t) = 0 \quad \forall t \in \mathbb{R}.$$

In each case we assume that the nonlinearities are given smooth functions.

Okazawa in [11] has studied some properties on $A + B$, where *A* is a linear operator with domain $D(A)$ That is dense in Banach space *X*, and *B* be a linear operator in *X*, with $D(B) \supset D(A)$ and presented by some certain conditions on *A* and *B*, if *A* is *m*-accretive and *B* is accretive then $A + B$ is also *m*-accretive, i.e., if –*A* is the generator of a contraction semigroup on *X* then so is $-(A + B)$, to.

Let *A* bea closed densely define linear map in a Banach space with infinite dimensional null space and *F* be a nonlinear map such that $A - F$ is pseudo *A* – proper. The solvability theory for the equation

$Ax - F(x) = f$

using only the Brouwer degree theory and the finite dimensional Morse theory has been presented and studied in [8].

The solvability of the equation $Au - Tu + Cu = f$ is studied under various assumptions of monotonicity and compactness on the operators *A*, *T* and *C*, which maps subsets of reflexive Banach space *X* in to dual space, [5].

Mortici in [9], [10] the semilinear equation $Au + F(u) = 0$, was considered, where $A : D(A) \subseteq H \to H$ is a linear maximal monotone operator and the nonlinear operator $F : H \to H$ is a (strongly monotone and Lipschitz operator) or quasi-positive. It is proved that, under these assumptions, the equation $Au + F(u) = 0$, has a unique solution.

Teodorescu in [16], had established an existence and uniqueness result for the equation $u - Au + F(u) = f$ where *A* is linear, quasi-positive and the nonlinear function *F* is a Lipschitz monotone operator.

Teodorescu in [17], the semilinear equation $Au + F(u) = f,$ was considered, where $A : D(A) \subseteq H \to H$ is a linear maximal monotone operator and the nonlinear operator $F : H \to H$ is a Lipschitz operator. It is proved that, under these assumptions, the equation $Au + F(u) = f$ has a unique solution.

Kdudhair in [6], established an existence and uniqueness result of semilinearequations $Ax + J(x) + F(x) = f$, where $A : D(A) \subseteq H \to H$ is a linear maximal monotone, strongly positive operator, $J : H \to H$ is a duality map and *F* is a Lipschitz operator.

In this paper, we focus on study the types of densely define unbounded linear operator is self-adjoint or generator of nonexpansive perturbed semigroup and suggested the basic condition for solving semilinear perturbed unbounded operator equations properties. The solvability of some such classes with perturbation dependent on the some estimator establish on the values of the resolvent set and using different type of fixed point theorems, such as Leray-Schauder mapping degree have been presented.

Preliminaries:

In this section, we review some basic definitions and theorems that we needed later on.

Definition (2.1), [1]:

A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space *X* is called a one-parameter semigroup$\mathbb{R}^+ \ni t \longrightarrow T(t)$ on *X*if it satisfies the following conditions:

1. $T(t+s) = T(t)T(s), \forall t, s \geq 0$
2. $T(0) = I$, (*I*is the identity operator on *X*).

A semigroup of bounded linear operator, *T*(*t*), is:
uniformly continuous if $\lim_{t\downarrow 0} \|T(t) - I\| = 0.$

Strongly continuous(C_0 -semigroup) if $\lim_{t\downarrow 0} \|T(t)x - x\|_X = 0$ for every $x \in X$.

Definition (2.2), [1]:

The linear operator *A* defined on the domain:

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\} \text{ and}$$

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)}{dt}\right|_{t=0} \quad \text{for } x \in D(A)$$

is the generator of the semigroup$T(t)$, *D*(*A*) is the domain of *A*.

Remark (2.1), [12]:

Let $T(t)$ be $C_\circ$-semigroup and let *A* be its generator, then:

i. $A : D(A) \subseteq X \to X$, is a linear generator.

ii. For $x \in X$

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\ ds = T(t)x,\quad h\in(0,t)$$

For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x$$

$= T(t)Ax$, for all $t \geq 0$.

Remark (2.2), [12]:

Let $T(t)$ be a C_0-semigroup then there are constants $w \geq 0$ and $M \geq 1$, such that $\|T(t)\| \leq Me^{wt}$ for $t \geq 0$. If $w = 0, T(t)$ is called uniformly bounded moreover if $M = 1$ it is called a C_0-semigroupof contractions.

Theorem (2.1), [12]:

A linear operator *A* is the generator of a uniformly continuous semigroup if and only if *A* is a bounded linear operator.

Definition (2.3), [4]:

Let *X* and *Y* be two normed spaces and *M*be a subspace of *X*, then the linear operator *T* defined on *M* into *Y* is called closed linear operator if for every convergent sequence $\{x_n\}$ of points of *M* with the limit *x*∈*X*, such that the sequence $\{Tx_n\}$ is convergent sequence with the limit *y*,*x*∈*M* and $y = Tx$.

We define the duality set $F(x) \in X^*$ by

$$F(x) = \{x^* : x^* \in X^* and \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

Definition (2.4), [12]:

A linear operator *A* is dissipative if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that Re$\langle Ax, x^*\rangle \leq 0$.

Lemma(2.1), [12]:

Let *X* be a Banach space and A generates a strongly continuous semigroup of contractions $\{T(t)\}_{t\geq 0}$ and $B : D(B) \subseteq H \to H$ be a dissipative which satisfies $D(A) \subset D(B)$ and $\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|$, for $x \in D(A)$ where $0 \leq \alpha < 1$ and $\beta \geq 0$.

Then $A+B:D(A)\subseteq H\to H$ is the generator of strongly continuous semigroup of contractions.

Lemma(2.2), [12]:

Let*X* be a reflexive Banach space and A generates a strongly continuous semigroup of contractions $\{T(t)\}_{t\geq 0}$ and $B:D(B)\subseteq H\to H$ be a dissipative which satisfies $D(A)\subset D(B)$ and $\|Bx\|\leq\|Ax\|+\beta\|x\|$ for $x\in D(A)$ where $\beta\geq 0$. Then $\overline{(A+B)}$, the closure of $A+B$, is the generator of strongly continuous semigroup of contractions.

$A+B:D(A)\subseteq H\to H$ be unbounded perturbed linear operator of strongly continuous semigroup of contractions.

Lemma (2.3) "Hille-Yosida", [12]:

A linear (unbounded) operator *A* is the generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ if and only if

1. *A* is closed and $\overline{D(A)}=X$.
2. The resolvent set $\rho(A)$ of *A* contains $\mathbb{R}^+$ and for every $\lambda>0$

$$\|(\lambda I-A)^{-1}\|_{L(X)}\leq\frac{1}{\lambda}.$$

Lemma (2.4), [12]:

A linear operator *A* is the generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\|_{L(X)}\leq Me^{wt}$, $M\geq 1, w\geq 0$ if and only if:

i. *A* is closed $\overline{D(A)}=X$.

ii. The resolvent set $\rho(A)$ of *A* contains the ray $\{\lambda:\ \operatorname{Im}\lambda=0,\ \lambda>w\}$,

$$\left\|\left((\lambda I-A)^{-1}\right)^n\right\|_{L(X)}\leq\frac{M}{(\lambda-w)^n},\ \text{for } \lambda>w\ ,\ n=1,2,\ldots$$

is bounded linear operator in *X*, where $\operatorname{Im}\lambda$ is standing for imaginary part of a complex eigenvalue.

Theorem (2.2), [12]:

Let *X* be a Banach space and *A* be the unbounded linear generator of a $C_\circ$-semigroup $T(t)$ on *X*, satisfying:

$$\|T(t)\|_{L(X)}\leq Me^{wt}.$$

If *B* is a bounded linear operator on *X*, then $A+B$ with $D(A+B)=D(A)$ is the generator of a perturbed $C_\circ$- semigroup $S(t)$ on *X*, satisfying:

$\|S(t)\|_{L(X)} \leq Me^{(w+M\|B\|)t}$.

Definition (2.5) "Adjoint operator of unbounded operator", [14]:

Let A be an unbounded linear operator on a Hilbert space *H* and assume that the domain of A , $D(A)$ is dense in *H*. Then the adjoint operator *A** of A is defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for all } x \in D(A), y \in D(A^*), \text{ where:}$$

$$D(A^*) = \{y \in H \,|\, \langle Ax, y\rangle = \langle x, z\rangle, \text{ for some } z \in H \text{ and all } x \in D(A)\}.$$

Definition (2.6) "Self-adjoint operator", [14]:

Let $A : D(A) \subset H \to H$ be densely defined linear operator in Hilbert space *H*, then *A* is called self-adjoint operator if $A = A^*$.

Theorem (2.3), [4]:

Let $A : D(A) \to H$ be a self adjoint linear operator, which is densely defined in a complex Hilbert space *H*. Then a number λ belongs to the resolvent set $\rho(A)$ of *A* if and only if there exists a $c > 0$ such that for every $x \in D(A)$,

$\|(\lambda I - A)x\| \geq c\|x\|$.

Definition (2.7) "Compact Operator", [19]:

Let *X* and *Y* be normed spaces, the operator $A : D(A) \subseteq X \to Y$ is called a compact if

1. *A* is continuous, and
2. *A* transforms bounded subset *M* of *X* into relatively compact subset in *Y* ($\overline{A(M)}$ is compact).

Lemma (2.5) "Compactness of Product", [4]:

Let $A : X \to X$ be a compact linear operator and $B : X \to X$ a bounded linear operator on a normed space *X*. Then *AB* and *BA* are compact.

Lemma (2.6) "Compactness Perturbation", [18]:

Let $\mathrm{f}, \mathrm{g}, \mathrm{h} : X \to X$ be mappings on a Banach space *X*, then f is called compact perturbation of the mapping g if and only if $\mathrm{f} = \mathrm{g} + \mathrm{h}$ and h is compact.

Definitions (2.8), [2]:

Let $A : D(A) \subset X \to Y$, where *X*, *Y* are Banach spaces. *A* is said to be completely continuous if for every sequence $\{x_n\} \subset D(A)$ such that $x_n \xrightarrow{w} x \in D(A)$, we have $Ax_n \longrightarrow Ax$.

Lemma (2.7), [14]:

1. Let X be a normed space, the mapping $A: D(A) \subseteq X \to X$ is called completely continuous if it is both continuous and compact.
2. A bounded set of reflexive Banach space is called weakly compact if every sequence has a converges weakly subsequence,

Remarks (2.3):

1. Let *X* and *Y* be normed spaces, then:

i. Every compact linear operator $B: X \to Y$ is bounded.

ii. If $\dim X = \infty$, then the identity operator $I: X \to X$ is not compact.

iii. If $\dim X < \infty$, then the identity operator $I: X \to X$ is compact, [4].

2. Let *X* and *Y* be Banach spaces and $A: X \to Y$ be an operator:

i. If *A* is a compact linear operator, then *A* is completely continuous.

ii. If *X* is reflexive and *A* (need not be linear) is completely continuous then *A* is compact, [3].

3. If *A* and*B* are completely continuous operators, then $\alpha_1 A + \alpha_2 B$ is also a completely continuous operator, where α_1, α_2 are scalar values.

Definition (2.9), [18]:

Let *X* and *Y* be real Hilbert space, and let $A: X \to X^*$ be an operator. Then

1. *A* is called monotone if and only if
$\langle Au - Av, u - v \rangle \geq 0$ for all $u, v \in X$.

2. *A* is called strictly monotone if and only if
$\langle Au - Av, u - v \rangle > 0$ for all $u, v \in X$ with $u \neq v$.

3. *A* is called strongly monotone if and only if there is a constant $c > 0$, such that
$\langle Au - Av, u - v \rangle \geq c \|u - v\|^2$ for all $u, v \in X$.

Remark (2.4), [18]:

Let$A: X \to X^*$ be a linear operator on real Hilbert space *X*, then:

A is strictly monotone if and only if *A* is strictly positive
(i.e., $\langle Au, u \rangle > 0$, for all $u \in X$ with $u \neq 0$).

A is strongly monotone if and only if *A* is strongly positive
(i.e., $\langle Au, u \rangle \geq c \|u\|^2$, for all $u \in X$ and fixed $c > 0$).

A is called coercive if and only if

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} \to +\infty \cdot$$

Definition (2.10), [18]:

Let $A: D(A) \subseteq H \to H$ be an operator on real Hilbert space*H*.

A is called monotone if and only if $\langle Au - Av, u - v \rangle \geq 0$ for all $u, v \in D(A)$.

1. *A* is called maximal monotone if and only if *A* is monotone and $\langle b - Av, u - v \rangle \geq 0$ for all $v \in D(A)$

Implies $Au = b$, i.e., *A* has no proper monotone extension.

2. *A* is called accretive if and only if $(I + \mu A): D(A) \to H$ is injective and $(I + \mu A)^{-1}$ is nonexpansive for all $\mu > 0$.
3. *A* is called maximal accretive (*m*-accretive)if and only if *A* is accretive and $(I + \mu A)^{-1}$ exists on *H* for all $\mu > 0$.

Lemma (2.8), [18]:

Let $A: D(A) \subseteq H \to H$ be an operator on the real Hilbert space*H*. Then the following three properties of *A* are mutually equivalent:

1. *A* is monotone and R(*I* + *A*) = *H*.
2. *A* is maximal accretive.
3. *A* is maximal monotone.

Lemma (2.9),[11]:

Let *A* be a linear operator with domain*D*(*A*) and range*R*(*A*) in a Banach space *X*. Let *B* be a linear operator in *X*, such that $D(A) \subset D(B)$ and $\|Bx\| \leq \alpha \|Ax\| + \beta \|x\|$ for $x \in D(A)$ where $0 \leq \beta$ and $1 > \alpha \geq 0$.If *A* is maximal accretive with *D*(*A*) dense in *X*, and *B* is accretive then$A + B$ is maximal accretive.

Definition (2.11), [18]:

The semigroup$T(t): H \to H$ of linear bounded operator is nonexpansivesemigroup if $T(t)$ is strongly continuous and $\|T(t)\| \leq 1$ for all $t \geq 0$.

Theorem (2.4), [18]:

Let $A: D(A) \subseteq H \to H$ be a linear operator on the real Hilbert space *H*with $\overline{D(A)} = X$. Then the following conditions are mutually equivalent:

1. *A* is the generator of a linear nonexpansivesemigroup.
2. $-A$ maximal accretive and $\overline{D(A)} = X$.

Definition (2.12) "Leray-Schauder type (LS)", [2]:

The operator of the type$I - T$ in infinite dimensional space, with *I* the identity mapping and *T* compactness called compact displacements of the identity.

Lemma (2.10), [10]:

Let $T : D \subset X \to X$, $T \in (LS)$ be such that $I - T$ is compact and $y \in X \setminus T(\partial D)$. Then the Leray-Schauder degree $\deg(T, D, y)$ satisfies the following properties:

1. If $\deg(T, D, y) \neq 0$, then $y \in T(D)$.
2. If $H \in C([0,1] \times D, X)$ is such that $I - H(t, \cdot)$ is compact, for all $t \in [0,1]$ and $y \in X \setminus H([0,1] \times \partial D)$, then the degree $\deg(H(t,\cdot), D, y) =$ constant, for all $t \in [0,1]$.
3. The degree for the identity map $I : X \to X$ is $\deg(I, D, y) = \begin{cases} 1, y \in D \\ 0, y \notin D \end{cases}$.

Solvability of Semilinear Perturbed Unbounded Class by Using Leray-Schauder Degree Theory

The proposed results of the following theorem depends on the results of Leray-Schauder degree, homotopy theorem of compact perturbation theorem and nonexpansive perturbed semigroup theory with some necessary conditions on perturbation operator, so the solvability is guaranteed , but the uniqueness is not necessary, so the existence of at least one solution is discussed.

In the following lemmas, we suggest important assumptions which make sense in some useful properties to develop and generalize the results later on.

Lemma (3.11):

Let $\{S(t)\}_{t \geq 0}$ be a $C_\circ$-perturbed semigroup of bounded linear operators generator by perturbed unbounded linear operator $(A + B)$, with $D(A) \subset D(B)$. Then $((\lambda - \|B\|)I - (A + B))$ is invertible for all $\lambda > \|B\|$ and $x \in D(A)$.

Proof:

Applying Laplace transform for $e^{\|B\|t} S(t)$, yield

$$F(\lambda)x = L(e^{\|B\|t} S(t)x) = \int_0^\infty e^{-\lambda t} e^{\|B\|t} S(t)x\,dt \text{ ,for } \lambda > \|B\| \text{ and } x \in X . \qquad (3.1)$$

$$\frac{S(h) - I}{h} F(\lambda)x = \frac{S(h) - I}{h} \int_0^\infty e^{-\lambda t} e^{\|B\|t} S(t)x\,dt \text{ , for } h > 0.$$

$$= \frac{1}{h} \int_0^\infty e^{-(\lambda - \|B\|)t} \left(S(h)S(t)x - S(t)x \right) dt$$

$$= \frac{1}{h} \int_0^\infty e^{-(\lambda - \|B\|)t} \left(S(t+h)x - S(t)x \right) dt$$

$$=\frac{1}{h}\int_0^\infty e^{-(\lambda-\|B\|)t}S(t+h)x\,dt-\frac{1}{h}\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)x\,dt$$

Let $\delta=t+h\Rightarrow d\delta=dt$, if $0\le t\le\infty$ then $h\le\delta\le\infty$, we get

$$\frac{S(h)-I}{h}F(\lambda)x=\frac{1}{h}\int_h^\infty e^{-(\lambda-\|B\|)(\delta-h)}S(\delta)x\,d\delta-\frac{1}{h}\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)x\,dt$$

$$=\frac{e^{(\lambda-\|B\|)h}}{h}\int_0^\infty e^{-(\lambda-\|B\|)\delta}S(\delta)x\,d\delta-\frac{e^{(\lambda-\|B\|)h}}{h}\int_0^h e^{-(\lambda-\|B\|)\delta}S(\delta)x\,d\delta$$

$$-\frac{1}{h}\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)x\,dt$$

We get

$$\frac{S(h)-I}{h}F(\lambda)x$$

$$=\frac{e^{(\lambda-\|B\|)h}-1}{h}\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)x\,dt-\frac{e^{(\lambda-\|B\|)h}}{h}\int_0^h e^{-(\lambda-\|B\|)t}S(t)x\,dt. \qquad (3.2)$$

As $h\downarrow 0,$ from (3.1) and remarks (2.1)(ii), the right-hand side of (3.2) converges to $(\lambda-\|B\|)F(\lambda)x-x$. Since the left-hand side is the generator of $S(t), t\ge 0$, this implies that for every $x\in X$ and $\lambda>\|B\|$,

$F(\lambda)x\in D(A+B)=D(A)$ and

$(A+B)F(\lambda)=(\lambda-\|B\|)F(\lambda)-I$, or

$$\big((\lambda-\|B\|)I-(A+B)\big)F(\lambda)=I\ . \qquad (3.3)$$

For $x\in D(A+B)=D(A)$, we have

$$F(\lambda)(A+B)x=\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)(A+B)x\ dt\ . \qquad (3.4)$$

From remarks (2.1)(iii), the Equation (3.4) become,

$$F(\lambda)(A+B)x=\int_0^\infty e^{-(\lambda-\|B\|)t}(A+B)S(t)x\ dt$$

$$=(A+B)\left(\int_0^\infty e^{-(\lambda-\|B\|)t}S(t)x\,dt\right)$$

$$=(A+B)F(\lambda)x. \qquad (3.5)$$

From (3.3) and (3.5) it follows that

$$F(\lambda)\big((\lambda-\|B\|)I-(A+B)\big)x=Ix\ ,\quad \text{for every } x\in D(A+B) \qquad (3.6)$$

Thus, $F(\lambda)$ is the inverse of $((\lambda-\|B\|)I-(A+B))$, it exists for all $\lambda>\|B\|$.

Lemma (3.12):

Let $\lambda I-(A+B),-\|B\|I$ are linear operators from *X* to *Y* (*X, Y* are two Hilbert spaces), with $\mathrm{D}(A)\subset\mathrm{D}(B)$ and $\lambda>\|B\|$, the linear operator $-\|B\|I$ is bounded and *F* denotes the duality map on *Y* to *Y** defined by for every $y\in Y$, $F(y)=\left\{g\in Y^*;\langle y,g\rangle=\|y\|^2=\|g\|^2\right\}$, Assume that for every $x\in D(\lambda I-(A+B))$ there is $g\in F((\lambda I-(A+B))x)$ such that,

$$\langle(-\|B\|I)x,g\rangle\ge -c\|x\|^2-a\|(\lambda I-(A+B))x\|\|x\|-b\|(\lambda I-(A+B))x\|^2 \quad (3.7)$$

Where

a,b and *c* are nonnegative constants, with

$c\left\|(\lambda I-(A+B))^{-1}\right\|^2+a\left\|(\lambda I-(A+B))^{-1}\right\|+b<1,$ and

$\left\|(\lambda I-(A+B))^{-1}\right\|^{-1}>0.$ Then $\left((\lambda-\|B\|)I-(A+B)\right)$ is invertible and $\left((\lambda-\|B\|)I-(A+B)\right)^{-1}\in B(Y,X)$, With

$$\left\|\left((\lambda-\|B\|)I-(A+B)\right)^{-1}\right\|\le\frac{1}{\lambda-\|B\|},\qquad \text{for all } \lambda>\|B\|. \quad (3.8)$$

Proof:

From theorem (2.2) and lemma (2.4), the $(\lambda I-(A+B))$ is invertible, that is $N(\lambda I-(A+B))=\{0\}$,

Then

$\|(\lambda I-(A+B))x\|\ge\left\|(\lambda I-(A+B))^{-1}\right\|^{-1}\|x\|$ for all

$x\in D(\lambda I-(A+B))$, $\left\|(\lambda I-(A+B))^{-1}\right\|^{-1}>0.$

Thus, $\|x\|\le\left\|(\lambda I-(A+B))^{-1}\right\|\|(\lambda I-(A+B))x\|$,

It follows from (3.7) that

$$\langle(-\|B\|I)x,g\rangle\ge -c\left\|(\lambda I-(A+B))^{-1}\right\|^2\|(\lambda I-(A+B))x\|^2$$

$$-a\left\|(\lambda I-(A+B))^{-1}\right\|\|(\lambda I-(A+B))x\|^2-b\|(\lambda I-(A+B))x\|^2$$

$$\ge(-c\left\|(\lambda I-(A+B))^{-1}\right\|^2-a\left\|(\lambda I-(A+B))^{-1}\right\|-b)\|(\lambda I-(A+B))x\|^2$$

$$\ge-(c\left\|(\lambda I-(A+B))^{-1}\right\|^2+a\left\|(\lambda I-(A+B))^{-1}\right\|+b)\|(\lambda I-(A+B))x\|^2. \quad (3.9)$$

From definition of duality we have that,

$$\|(\lambda I-(A+B))x\|^2=\langle(\lambda I-(A+B))x,g\rangle. \quad (3.10)$$

Thus,

$$\langle\left((\lambda-\|B\|)I-(A+B)\right)x,g\rangle=\langle(\lambda I-(A+B))x,g\rangle+\langle(-\|B\|I)x,g\rangle$$

From (3.9) and (3.10), we have that

$$\langle\big((\lambda-\|B\|)I-(A+B)\big)x,g\rangle=\|(\lambda I-(A+B))x\|^2+\langle(-\|B\|I)x,g\rangle$$

$$\geq\|(\lambda I-(A+B))x\|^2-(c\|(\lambda I-(A+B))^{-1}\|^2+a\|(\lambda I-(A+B))^{-1}\|+b)\|(\lambda I-(A+B))x\|^2$$

$$\geq[1-c\|(\lambda I-(A+B))^{-1}\|^2-a\|(\lambda I-(A+B))^{-1}\|-b]\|(\lambda I-(A+B))x\|^2. \quad (3.11)$$

Therefore, from definition of duality, we obtain

$$\|\lambda I-(A+B)x\|\leq\frac{1}{\sqrt{1-c\|(\lambda I-(A+B))^{-1}\|^2-a\|(\lambda I-(A+B))^{-1}\|-b}}\|(\lambda I-(A+B)-\|B\|I)x\|$$

$$\left[\sqrt{1-c\|(\lambda I-(A+B))^{-1}\|^2-a\|(\lambda I-(A+B))^{-1}\|-b}\right]\|(\lambda I-(A+B))\|\leq\|(\lambda I-(A+B)-\|B\|I)\|$$

$$\frac{1}{\left[\sqrt{1-c\|(\lambda I-(A+B))^{-1}\|^2-a\|(\lambda I-(A+B))^{-1}\|-b}\right]\|(\lambda I-(A+B))\|}\geq\frac{1}{\|(\lambda I-(A+B)-\|B\|I)\|}$$

From lemma (3.11), we have that $\big((\lambda-\|B\|)I-(A+B)\big)$ is invertible linear operator and $\big(\lambda I-(A+B)\big)$ is invertible [see theorem (2.2) and lemma (2.4)], hence

$$\|((\lambda-\|B\|)I-(A+B))^{-1}\|\leq\left[\sqrt{1-c\|(\lambda I-(A+B))^{-1}\|^2-a\|(\lambda I-(A+B))^{-1}\|-b}\right]^{-1}\|(\lambda I-(A+B))^{-1}\|$$

Since *a*, *b* and *c* are nonnegative, if we set $a=b=c=0$, then we have that

$$\left\|\big((\lambda-\|B\|)I-(A+B)\big)^{-1}\right\|\leq\|(\lambda I-(A+B))^{-1}\|\leq\frac{1}{\lambda-\|B\|},\text{ for all }\lambda>\|B\|.$$

Let $\mu=\lambda-\|B\|$ then $\left\|\big(\mu I-(A+B)\big)^{-1}\right\|\leq\frac{1}{\mu}$, for all $\mu>0$. (3.12)

Concluding Remark (3.1):

Let *X* be a Banach space and let *A* be the generator of $C_\circ$-semigroup of contraction (nonexpansivesemigroup) in *X*. *B* is a bounded linear operator perturbation to *A* such that $D(A)\subset D(B)$ and (3.7) holds for a,b and *c* are nonnegative constants, with

$c\|(\lambda I-(A+B))^{-1}\|^2+a\|(\lambda I-(A+B))^{-1}\|+b<1$, and $\lambda>\|B\|$. Then $A+B$, is the generator of $C_\circ$-semigroup of contraction (nonexpansivesemigroup) in *X*.

The following lemma is important in the following result and its proof is helping us to go further, so it has been imposed for benefit point of view.

Lemma (3.13), [10]:

If $F:H\rightarrow H$ is a quasi-positive operator with $\alpha>1/2$, then

$\|x - F(x)\| \leq \|x\|$, for all $x \in H, x \neq 0.$

Theorem (3.5):

Let H be a real Hilbert space, $A + B : D(A) \subseteq H \to H$ be perturbed unbounded linear operator generator of perturbed $C_\circ$-semigroup of $\{S(t)\}_{t \geq 0}$, where A is a generator of $C_\circ$-semigroup of contraction $\{T(t)\}_{t \geq 0}$ and $B : D(B) \subseteq H \to H$ be a bounded linear operator perturbation to A such that $0 \in \operatorname{Int} D(A)$, $D(A) \subset D(B)$, $\lambda > \|B\|$ and satisfy for every $x \in D(\lambda I - (A + B))$ there is $g \in F((\lambda I - (A + B))x)$

$\langle (-\|B\| I)x, g\rangle \geq -c\|x\|^2 - a\|(\lambda I - (A + B))x\|\|x\| - b\|(\lambda I - (A + B))x\|^2$,

and $F : H \to H$ be a nonlinear operator of Leray-Schauder type operator such that $\langle F(x), x\rangle \geq \alpha \|F(x)\|^2$, $\alpha > 1/2$ for all $x \in H, x \neq 0.$

Then the semilinear perturbed unbounded operator equations

$$F(x) = (A + B)x. \qquad (3.13)$$

has at least one solution $x \in D(A).$

Proof:

From concluding remark (3.1), we have that $(A + B)$ is a generator of nonexpansivesemigroup of bounded linear operators $\{S(t)\}_{t \geq 0}$, thus, by using theorem (2.4), we have that $-(A + B)$ is a maximal accretive.

Our aim to examine the existence solution of the following system

$$-(A + B)x + F(x) = 0. \text{ for } x \in D(A). \qquad (3.14)$$

From lemma (2.8), we have that $-(A + B)$ is a unique maximal monotone on *H*. By using lemma (3.12), we get for all $\mu > 0$ and $\frac{1}{\mu} \in \rho(A + B)$ the operator $(I - \mu(A + B))$ is invertible with continuous inverse

$$(I - \mu(A + B))^{-1} : H \to H \text{ and } \|(I - \mu(A + B))^{-1}\| \leq 1. \qquad (3.15)$$

Suppose that $\mu = 1.$

Now, the Equation (3.14) can be written as

$$(I - (A + B))x + (-I + F)(x) = 0$$

$$(I - (A + B))x = (I - F)(x)$$

From (3.15), we obtain that

$$x = (I - (A + B))^{-1}(I - F)(x),$$

Or $\quad x = T(x) \Leftrightarrow (I - T)(x) = 0, \qquad (3.16)$

Where

$$T = (I - (A + B))^{-1}(I - F).$$

Since F is an operator of Leray-Schauder type, from definition (2.12), we get $I-F$ is compact and since $(I-(A+B))^{-1}$ is continuous. By lemma (2.5), hence $T=(I-(A+B))^{-1}(I-F)$ is a compact operator.
one can get

$$(I-T)=(I-(I-(A+B))^{-1}(I-F)), \qquad (3.17)$$

is a compact perturbation of the identity map by definition (2.12) and from (3.16), hance

$$(I-(I+(A+B))^{-1}(I-F))x=0. \qquad (3.18)$$

Can be solved using the Leray-Schauder degree [see lemma (2.10)] as follows:
Since

$$-(A+B)x+F(x)=0 \Leftrightarrow x=(I-(A+B))^{-1}(I-F)(x)$$
$$\Rightarrow (I-(I-(A+B))^{-1}(I-F))x=0$$

Let $B=B(0,r)$ such that $\overline{B(0,r)}\subset \mathrm{D}(A+B)$. By using Leray-Schauder, we have that

$$\mathrm{H}(t,x)=x-t(I-(A+B))^{-1}(I-F)(x),\ x\in\overline{\mathrm{B}},\ t\in[0,1]. \qquad (3.19)$$

If $0\in \mathrm{H}(1,\partial\mathrm{B}),$ the conclusion follows immediately. In order to use the invariance to homotopy of the Leray-Schauder degree, we prove that $0\notin \mathrm{H}([0,1],\partial\mathrm{B}).$

Let us suppose by contrary that $\mathrm{H}(t,x)=0,$ for some $x\in\partial\mathrm{B}$ and $t\in[0,1)$.
From equation (3.19) It results

$$\|x\|=t\left\|(I-(A+B))^{-1}(I-F)(x)\right\|$$
$$\le \left\|(I-(A+B))^{-1}(I-F)(x)\right\|$$
$$\le \left\|(I-(A+B))^{-1}\right\|.\left\|x-F(x)\right\|,$$

By (3.15) and lemma (3.13) we get

$$\|x\|\le\|x-F(x)\|\le\|x\|$$

we must have equalities all over, in particular $(I-(A+B))^{-1}(I-F)(x)=0.$ hence $x=0\in\partial\mathrm{B},$ contradiction. This means that $0\notin \mathrm{H}([0,1],\partial\mathrm{B}).$ and further, by lemma (2.10)

$$\mathrm{d}(\mathrm{H}(1,\cdot),\mathrm{B},0)=\mathrm{d}(\mathrm{H}(0,\cdot),\mathrm{B},0)\Rightarrow$$
$$\Rightarrow \deg(I-(I+(A+B))^{-1}(I-F)(x),\mathrm{B},0)=\deg(I,\mathrm{B},0)=1.$$

Hence, $\deg(I-(I+(A+B))^{-1}(I-F)(x),\mathrm{B},0)\neq 0$.
By Leray-Schauder degree, one gets:
The Equation (3.18)and equivalent, the Equation (3.14), has at least one solution in $D(A)$. Hence $F(x)=(A+B)x$ has at least one solution in $D(A)$.

Concluding remark (3.2):

The selection of λ in theorem (3.4), will not effect the solvability result. The solution is independent of chose of λ.

Corollary (3.1):

Let $A : D(A) \subseteq H \to H$ be a generator of nonexpansivesemigroup $\{T(t)\}_{t\geq 0}$ and $B : D(B) \subseteq H \to H$ be a dissipative which satisfies $D(A) \subset D(B)$, and $\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|$ for $x \in D(A)$, where $0 \leq \alpha < 1$ and $\beta \geq 0$.

Let $F : H \to H$ be a nonlinear operator is Leray-Schauder type operator such that $\langle F(x), x\rangle \geq \alpha\|F(x)\|^2, \alpha > 1/2$ for all $x \in H, x \neq 0$.

Then the semilinear perturbed unbounded operator equations

$$F(x) = (A + B)x. \tag{3.20}$$

has at least one solution in $D(A)$.

Proof:

According to lemma (2.1) and theorem (3.5), the proof is of direct way.□

Corollary (3.2):

Let $A : D(A) \subseteq H \to H$ be a generator of nonxpansivesemigroup $\{T(t)\}_{t\geq 0}$ and $B : D(B) \subseteq H \to H$ be a dissipative which satisfies $D(A) \subset D(B)$ and $\|Bx\| \leq \|Ax\| + \beta\|x\|$ for $x \in D(A)$ where $\beta \geq 0$. Let $F : H \to H$ be a nonlinear operator is Leray-Schauder type operator such that

$\langle F(x), x\rangle \geq \alpha\|F(x)\|^2$, $\alpha > 1/2$ for all $x \in H, x \neq 0$.

Then the semilinear perturbed unbounded closure operator equations

$$F(x) = \overline{(A + B)}x. \tag{3.21}$$

has at least one solution in $D(A)$.

Proof:

On the results of lemma (2.2) and theorem (3.5), the proof is of direct way.□

Illustrative example (3.1):

Let us consider the following semilinear heat flow equation

$$g(t, x(t)) = (\overline{\Delta} + \mu I)x(t). \tag{3.22}$$

where

the linear operator $\overline{\Delta}$ is the closure of the Laplace operator

$\Delta x(t) = \sum_{i=1}^{n} \frac{\partial^2}{\partial t_i^2} x(t_1, t_2, \ldots, t_n)$ defined for every x in the Schwartz space

$\delta(\mathbb{R}^n) = \{x \in C^\infty(\mathbb{R}^n) : \lim_{|t|\to\infty} |t|^n D^\alpha x(t) = 0$ for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n\}$

and assume that for $\mu > 0$ and for every $u \in \mathrm{D}(\lambda I - (\overline{\Delta} + \mu I))$ there is $g \in F((\lambda I - (\overline{\Delta} + \mu I))u)$ such that

$$\langle(-\mu I)u, g\rangle \geq -c\|x\|^2 - a\|(\lambda I - (\overline{\Delta} + \mu I))x\|\|x\| - b\|(\lambda I - (\overline{\Delta} + \mu I))x\|^2$$

where a,b and c are nonnegative constants and $\lambda > \mu$.

The particular case when $g(t,x) = a_0(t)x$, with $a_0 \in C(\overline{\Omega}), a_0 > p > 0$ is studied in [10], under the assumption that the nonlinear part is quasi-positive, we have that

$$\langle g(t,x(t)), x(t)\rangle = \int_\Omega g(t,x(t))x(t)dt$$

$$\geq \alpha \|g(t,x(t))\|^2 \quad \text{for some } \alpha > \frac{1}{2}$$

$$= \alpha \int_\Omega g^2(t,x(t))dt = \alpha \int_\Omega a_0^2(t)x^2 dt$$

$$= \alpha \|a_0\|^2 \|x\|^2. \tag{3.23}$$

On other hand, we have that

$$\langle g(t,x(t)), x(t)\rangle = \langle a_0(t)x(t), x(t)\rangle$$

By Cauchy Schwarz inquality, we obtain

$$\langle g(t,x(t)), x(t)\rangle \leq \|a_0\|\|x\|^2. \tag{3.24}$$

Thus from (2.55) and (2.56), we obtain

$$\|a_0\|\|x\|^2 \geq \alpha \|a_0\|^2 \|x\|^2 \tag{3.25}$$

Hence, $\alpha \|a_0\| \leq 1$.

Indeed, we can apply theorem (3.5) as follows;
$H = L^2(\mathbb{R}^n), (A+B)x = (\overline{\Delta} + \mu I)x$, generator a $C_\circ$-semigroup of contraction on $L^2(\mathbb{R}^n)$ with domain $D(A) = \delta(\mathbb{R}^n)$, and $(Fx)(t) = g(t,x)$.
The problem (3.22) can be written in the abstract form

$$F(x) = (A+B)x, \; x \in D(A) \subset L^2(\mathbb{R}^n).$$

□

The new results of the following theorem depend on the results of Leray-Schauder degree, homotopy theorem of compact perturbation theorem of nonlinear operators and some necessary conditions of a self-adjoint densely defined linear operator, so the solvability is guaranteed, but the uniqueness is not necessary, so the existenece of at least one solution is discussed.

Theorem (3.6):

Let *H* be a real Hilbert space. Let $A : D(A) \subset H \to H$ be an unbounded self-adjoint linear operator which is densely define in *H*. Let $F : H \to H$ be a semilinear operator such that $F = F_1 - F_2$,where

i. $F_1 : H \longrightarrow H$ is a compact linear operator;

ii. $F_2 : H \longrightarrow H$ is a completely continuous nonlinear operator.

Assume that there exists a completely continuous function

$g:\overline{B(0,1)}\longrightarrow \mathbb{R}^{+}$ such that $g(x)=0$ implies $x=0$. Furthermore there exists a constants $p>1, C>0$ such that

$\|(\lambda I-A)x\|\geq C\|x\|^{p+1}$, for every $x\in D(A)$. (3.26)

Then the following unbounded linear operator equations

$$Ax+F(x)=-f \qquad (3.27)$$

has at least one solution for any $f\in H$.

Proof:

Consider the following equation

$-Ax+\frac{1}{m}Ix-F(x)-f=0$, for any positive integer m. (3.28)

From theorem (2.3), we can define

$R_m=(\frac{1}{m}I-A)^{-1}:H\rightarrow D(A)$ for $\frac{1}{m}\in\rho(A)$ and R_m is linear bounded (hence continuous). Now (3.28) is equivalent to

$x-R_m\left(F(x)+f\right)=0$. (3.29)

By (i) we have that F_1 is a compact, and from remark ((2.3)(2)), we have that F_1 is a completely continuous, from concluding remark ((2.3)(3)) F_1-F_2 is a completely continuous semilinear operator, and from remark ((2.3)(2)), we have that F_1-F_2 is a compact semilinear operator.

Hence from lemma (2.5), we have $R_m\left(F(x)+f\right):H\longrightarrow D(A)$ is a compact semilinear operator.

By Leray-Schauder degree theory [see lemma (2.10)], Equation (3.29)is solvable if there exists $r>0$ such that $x-tR_m\left(F(x)+f\right)\neq 0$ for any $t\in(0,1]$, $x\in\partial B(0,r)$. Equivalently to show that the solutions of $x-tR_m\left(F(x)+f\right)=0$ for any $t\in(0,1]$ are uniformly bounded.

First to show that there exists $\varepsilon>0$ such that $g\left(x_t/\|x_t\|\right)>\varepsilon$ for any solution x_t of $x-tR_m\left(F(x)+f\right)=0$ with $t\in(0,1]$, $\|x_t\|\geq 1$.

Suppose that is not true, so there exists $\{t_n\}\subset(0,1]$ and $\{x_n\}\subset H$, $\|x_t\|\geq 1$, with

$x_n-t_nR_m\left(F(x_n)+f\right)=0$ and $g\left(x_n/\|x_n\|\right)\longrightarrow 0$ as $n\rightarrow\infty$.

By lemma((2.7)(2)) and H is a reflexive Hilbert space, $\overline{B(0,1)}$ is weakly compact. Thus, $\{x_j\}$ subsequence of $\{x_n\}$ such that $x_j/\|x_j\|\xrightarrow{w}x_0$ for some

$x_0 \in \overline{B(0,1)}$. Since g is completely continuous, $g\left(x_j/\|x_j\|\right) \longrightarrow g(x_0)$, thus, $g(x_0)=0$. We have $x_0=0$. Hence $x_j/\|x_j\| \xrightarrow{w} 0$.

Since $x_j - t_j R_m\left(F(x_j)+f\right)=0$, we have $\dfrac{x_j}{t_j} - R_m\left(F(x_j)+f\right)=0$,

thus

$$-A\frac{x_j}{t_j}+\frac{1}{m}I\frac{x_j}{t_j}-F(x_j)-f=0 \qquad (3.30)$$

Notice that $x_j/t_j \in D(A)$, since $R_m : H \to D(A)$. Thus,

$$\frac{1}{m}Ix_j = t_jA\frac{x_j}{t_j}+t_jF(x_j)+t_jf$$

$$=t_jA\frac{x_j}{t_j}+t_jF_1(x_j)-t_jF_2(x_j)+t_jf,$$

$$\left\langle \frac{1}{m}Ix_j, x_j\right\rangle = t_j\left\langle A\frac{x_j}{t_j}, x_j\right\rangle + t_j\left\langle F_1(x_j), x_j\right\rangle - t_j\left\langle F_2(x_j), x_j\right\rangle + t_j\left\langle f, x_j\right\rangle$$

$$\le t_j\left\langle A\frac{x_j}{t_j}, x_j\right\rangle + t_j\left\langle F_1(x_j), x_j\right\rangle + t_j\left\langle f, x_j\right\rangle,$$

which implies

$$\frac{1}{m}=\left\langle \frac{1}{m}\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\right\rangle \le t_j\left\langle A\frac{x_j}{t_j\|x_j\|}, \frac{x_j}{\|x_j\|}\right\rangle + t_j\left\langle F_1\left(\frac{x_j}{\|x_j\|}\right), \frac{x_j}{\|x_j\|}\right\rangle + t_j\left\langle f, \frac{x_j}{\|x_j\|}\right\rangle \frac{1}{\|x_j\|} \to 0$$

As $j\to\infty$, i.e., a contradiction. Hence, we have $g\left(x_t/\|x_t\|\right)>\varepsilon$ for some $\varepsilon>0$.

Now if $x_t - tR_m\left(F(x_t)+f\right)=0$ with $\|x_t\|\ge 1$, then

$$-A\frac{x_t}{t}+\frac{1}{m}I\frac{x_t}{t}-F(x_t)-f=0 \qquad (3.31)$$

and $$0=-\left\langle A\frac{x_t}{t}, x_t\right\rangle + \frac{1}{m}\left\langle I\frac{x_t}{t}, x_t\right\rangle - \left\langle F(x_t), x_t\right\rangle - \left\langle f, x_t\right\rangle$$

$$=\left\langle \left(\frac{1}{m}I-A\right)\frac{x_t}{t}, x_t\right\rangle - \left\langle F(x_t), x_t\right\rangle - \left\langle f, x_t\right\rangle$$

$$\geq \frac{C}{t}\|x_t\|^{p+2} - \|F\|\|x_t\|^2 - \|f\|\|x_t\|.$$

Since $p > 1$, $\|x_t\|$ must be a uniformly bounded. We get that (3.28) and (3.29) are solvable for any integer *m*.

Now, to show that the solutions of (3.28) are uniformly bounded with respect to *m*. suppose that is not uniformly bounded, let $\{x_n\}$ be a sequence such that $-Ax_n + \frac{1}{n}Ix_n - F(x_n) - f = 0$ and $\|x_n\| \to \infty$ as $n \to \infty$.

Assume that $\|x_n\| \geq 1$.

First we show that there exists $\varepsilon > 0$ such that $g\left(x_n/\|x_n\|\right) > \varepsilon$ for any $x_n \in \{x_n\}$.

If this is not true there exists a subsequence of $\{x_n\}$, such that $g\left(x_n/\|x_n\|\right) \to 0$. Since $\overline{B(0,1)}$ is weakly compact, we get as above $g(x_0) = \lim_{n\to\infty} g\left(x_n/\|x_n\|\right) = 0$, hence, $x_0 = 0$ so we have $x_n/\|x_n\| \xrightarrow{w} 0$.

Now,

$$F_1(x_n) = (\frac{1}{n}I - A)x_n + F_2(x_n) - f \ ,$$

$$\langle F_1(x_n), x_n\rangle \geq \left\langle (\frac{1}{n}I - A)x_n, x_n\right\rangle - \langle f, x_n\rangle \geq c\|x_n\|^{p+2} - \langle f, x_n\rangle,$$

where*c* some positive constant. So

$$\left\langle F_1(\frac{x_n}{\|x_n\|}), \frac{x_n}{\|x_n\|}\right\rangle \geq \frac{c\|x_n\|^{p+2}}{\|x_n\|^2} - \frac{1}{\|x_n\|}\left\langle f, \frac{x_n}{\|x_n\|}\right\rangle \geq c\|x_n\|^p.$$

The right side converge to $c\|x_n\|^p > 0$, and the left side converge to zero, we have $g\left(x_n/\|x_n\|\right) > \varepsilon$ for some $\varepsilon > 0$. Now

$$0 = -\langle Ax_n, x_n\rangle + \left\langle \frac{1}{n}Ix_n, x_n\right\rangle - \langle F(x_n), x_n\rangle - \langle f, x_n\rangle$$

$$= \left\langle (\frac{1}{n}I - A)x_n, x_n\right\rangle - \langle F(x_n), x_n\rangle - \langle f, x_n\rangle$$

$$\geq c\|x_n\|^{p+2} - \|F\|\|x_n\|^2 - \|f\|\|x_n\|$$

since $p > 1$ and $\|x_n\| \to \infty$, we have

$c\|x_n\|^{p+2} - \|F\|\|x_n\|^2 - \|f\|\|x_n\| \to \infty$ as $n \to \infty$. By this contradiction, we get that $\{x_n\}$ is uniformly bounded.

Now, since $\{x_n\}$ is a bounded sequence and *H* is a Hilbert space, then from lemma((2.7)(2)), we assume that $x_n \xrightarrow{w} x$ for some $x \in H$. Then $\frac{1}{n} x_n \to 0$ and since F_1 is a compact, then $F_1 x_n \to F_1 x$, $F_2(x_n) \to F_2(x)$ and from definition (2.6), *A* is a closed operator.

Hence, $\lim_{n\to\infty}(Ax_n + F(x_n) = -f\ ) \Rightarrow Ax + F(x) = -f$

has at least one solution for any $f \in H$. □

Concluding Remark(3.3):

The new results concluded from theorem (3.5), for perturbed linear operators that If $A+B$ is a generator of nonexpansivesemigroup,where *A* is a generator of nonexpansivesemigrouup and *B* linear bounded operator with some necessary conditions then $-(A+B)$ is a maximal accretive consisted from maximal accretive and bounded operators. In general, this is not true as example if *A* is a linear maximal accretive then it is maximal monotone, implies to monotone $\langle Ax, x\rangle \ge 0$ and $B = {}^{-}2A$ is abounded linear operator, then $\langle (A+B)x, x\rangle = \langle (A-2A)x, x\rangle = -\langle (Ax, x\rangle < 0,$ therefor *A*+*B* is not a maximal monotone and not maximal accretive.

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