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The Stability Analysis of Eco-Epidemiological System Involving a prey refuge

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ABSTRACT

In this research a predator–prey model involving disease in a prey and prey refuge has been proposed and analyzed. It is assumed that only the prey species is divided in to two classes infected and susceptible and disease transmitted by contact between a prey species. The existence, boundedness , permanence of the model has been investigated. The local and global stability conditions of all possible equilibrium points are established. Finally, numerical simulation is curried out to study the global dynamics of the model.

Keyword: Predator–prey model; prey refuge; local stability; global stability; Permanence.

تحليل الاستقرارية لنظام بيئي- وبائي والمتضمن ملجأ للفريسة

في هذا البحث تم اقتراح وتحليل نموذج المفترس والفريسة التي تتظمن على مرض في الفريسة وملجأ للفريسة .فمن المفترض أن فقط انواع الفريسة تنقسم إلى فئتين المصابة والسليمة والأمراض تنتقل عن طريق الاتصال بين انواع الفريسة .وقد تم التحقيق في الوجود، المحدودية و ديمومة النموذج .وضعت شروط الاستقرار المحلي والشامل من جميع نقاط التوازن الممكنة .أخيرا، يتم استخدام المحاكاة العددية لدراسة الديناميكية الشاملة للنموذج.

INTRODUCTION

Although, the dynamical behavior of the prey_predator model is well known ,and has taken a lot of interest since the pioneer work of classical Lotka-Volttera model [1,2], there are various ecological factors affect the existence and stability of this system such as prey refuge,disease, delay. harvesting and many other factors. Theoretical research and field Observations on population dynamics of prey refuges lead to the conclusion that the existence of prey refuges have stabilizing influences on prey-predator rnodels and prevent prey extinction due to predation [3,4,5]. Ruxton [6] proposed a continuous-time prey-predator model under the assumption that the rate prey moving to refuges is proportional to predator density and the results showed that the hiding behavior of prey has a stabilizing effect. The

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stabilizing effect also observed in a simple prey-predator system by Gonzalez-Olivares and Ramos-Jiliberto [7]. Ma et al. [8] formulated a prey-predator model with a class of functional response incorporating the effect of prey refuges and observed the stabilizing and destabilizing effect due to the increases in the prey refuges.

In this research however, prey-predator model with lotka-volttera functional response involving prey refuge has been proposed and analyzed. It is assumed that only the prey species is divided into two classes infected and susceptible. The dynamical behavior of the proposed system is investigated analytically as well as numerically The effect of prey refuge on the dynamical behavior of the system is discussed.

Mathematical Model:

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey-predator model involving prey refuge, in which $X(T)$ represents the prey population density at time T and $Y(T)$ that represents the density of the predator population at time T. It is assumed that the prey population is infected by *SIS*-type infectious disease. Further, the following assumptions are made in formulating the basic eco-epidemiological model:

In the absence of disease, the prey population grows logistically with carrying capacity $K > 0$ and intrinsic birth rate $r_1 > 0$. In the presence of disease, the prey population is divided into two groups, namely susceptible prey denoted by $S(T)$ and infected prey denoted by $I(T)$. Therefore at time T, the total population is $X(T) = S(T) + I(T)$ .It is assumed that the disease is not genetically inherited and the disease transmitted between the prey's individuals according the simple law of mass action with the transmission (infected) rate $\beta > 0$. In addition the infected prey don't has the capability of reproducing, however it still contribute with susceptible prey towards the carrying capacity of the system. Further the infected prey may recover and becomes susceptible again with the recover rate $c > 0$, while it faces death due to existence of disease with the disease death rate $d_1 > 0$.

The predator species consumes the susceptible and infected prey species according to the Lotka-Volttera functional response with the predation rates $b_1 > 0$ and $b_2 > 0$ respectively. However it gains food from them with the conversion rates $0 < e_1 < 1$ and $0 < e_2 < 1$ respectively. Finally in the absence of the prey species the predator decay exponentially with the death rate $d_2 > 0$. The prey species have a refuge protecting zone against the predation in the environment with refuge protection rates $0 \le n < 1$ and $0 \le m < 1$ of susceptible prey and infected prey respectively. Consequently this leave $(1-n)S$ of susceptible prey and $(1-m)I$ of infected prey are available to the predator.

According to the above hypotheses, the dynamics of a refuge prey-predator model with the disease in prey can be represented by the following set of differential equations.

$$\frac{dS}{dT} = r_1 S(1 - \frac{S+I}{K}) - \beta SI + cI - b_1(1-n)SY$$

$$\frac{dI}{dT} = \beta SI - cI - d_1 I - b_2(1-m)IY \qquad (1.1)$$
$$\frac{dY}{dT} = -d_2 Y + e_1 b_1 (1-n)SY + e_2 b_2 (1-m)IY$$

here
$S(T) \geq 0$, $I(T) \geq 0$ and $Y(T) \geq 0$. Clearly, system (1.1) has (12) parameters, which make the analysis difficult; to reduce the number of parameters and to determine which combination of parameters control the behavior of the system, the following non dimensional variables are used in system (1.1)

$$X_1 = \frac{S}{K}, X_2 = \frac{I}{K}, X_3 = \frac{Y}{K}, t = r_1 T$$

Now straightforward computation on system (1.1) gives the following dimensionless system

$$\frac{dX_1}{dt} = X_1(1 - X_1 - X_2) - \alpha_1 X_1 X_2 + \alpha_2 X_2 - \alpha_3(1-n)X_1 X_3$$
$$\frac{dX_2}{dt} = \alpha_1 X_1 X_2 - \alpha_4 X_2 - \alpha_2 X_2 - \alpha_5(1-m)X_2 X_3 \qquad (1.2)$$
$$\frac{dX_3}{dt} = -\alpha_6 X_3 + e_1 \alpha_3 (1-n) X_1 X_3 + e_2 \alpha_5 (1-m) X_2 X_3$$

where

$$\alpha_1 = \frac{K\beta}{r_1} > 0, \alpha_2 = \frac{c}{r_1} > 0, \alpha_3 = \frac{b_1 K}{r_1} > 0,$$
$$\alpha_4 = \frac{d_1}{r_1} > 0,\ \alpha_5 = \frac{b_2 K}{r_1} > 0,\ \alpha_6 = \frac{d_2}{r_1} > 0$$

represent the dimensionless parameters. It is observed that the dimensionless system (1.2) has ten parameters. Moreover, due to the biological meaning of the dependent variables $X_1(t), X_2(t)$ and $X_3(t)$ given in system (1.2), system (1.2) has the following domain $R_+^3 = \{(X_1, X_2, X_3), X_1 > 0, X_2 > 0, X_3 > 0\}$. Moreover, the interaction functions in the right hand side of system (1.2) are continuously differentiable on R_+^3 , and hence they are Lipschizian on R_+^3. Thus for each set of initial conditions, say $X_1(0) > 0, X_2(0) > 0, X_3(0) > 0$, system (1.2) has a unique solution. Therefore, the domain R_+^3 is an invariant set for the system (1.2). Further in the following theorem the uniform boundedness of the solution of system (1.2) is presented.

Theorem (1.1): All the solutions of system (1.2), which initiate in R_+^3 are uniformly bounded.

Proof: Let (X_1, X_2, X_3) be any solution of the system (1.2) with non negative initial condition $(X_1(0), X_2(0), X_3(0))$. Assume that $W(t) = X_1 + X_2 + X_3$, then

$$\frac{dW}{dt} = X_1(1-X_1-X_2) - \alpha_3(1-n)X_1X_3(1-e_1)$$
$$-\alpha_4X_2 - \alpha_6X_3 - \alpha_5(1-m)X_2X_3(1-e_2)$$
$$\frac{dW}{dt} \le X_1 - \alpha_4X_2 - \alpha_6X_3$$
$$\frac{dW}{dt} \le 2X_1 - \delta W\,, \quad \delta = \min.\{1, \alpha_4, \alpha_6\}$$

Now since the prey species growth logistically, which is bounded by its carrying capacity, then from the first equation on system (1.2) its easy to show that

$$\frac{dX_1}{dt} \le X_1(1-X_1)$$

from which we get that $Sup.X_1 \le \widehat{K}$, $\forall\, t > 0$ where $\widehat{K} = \max.\{X_1(0), 1\}$. substituting this in the above equation we obtain

$$\frac{dW}{dt} \le 2\widehat{K} - \delta W$$

from which its obtained that $0 \le W \le \frac{2\hat{K}}{\delta}$, as $t \to \infty$. Hence all the solutions of system (1.2) that initiate in R_+^3 are confined in the region $B = \{(X_1, X_2, X_3) : 0 \le W \le \frac{2\hat{K}}{\delta} + \varepsilon$ for any $\varepsilon > 0\}$

Existence of Equilibrium Points:

System (1.2) has at five biologically feasible equilibrium points. The existence conditions for each of these equilibrium points are discussed in the following:

1. Trivial equilibrium point $E_0 = (0,0,0)$ always exists.
2. Axial equilibrium point $E_1 = (1,0,0)$ always exists.
3. Predator free equilibrium point $E_2 = (\hat{X}_1, \hat{X}_2, 0)$, where

$$\widehat{X}_1 = \frac{\alpha_4 + \alpha_2}{\alpha_1}, \widehat{X}_2 = \frac{\alpha_4 + \alpha_2}{\alpha_2\alpha_4 + \alpha_2 + \alpha_4}\left[1 - \frac{\alpha_4 + \alpha_2}{\alpha_1}\right] \qquad (2.1)$$

exists uniquely in $Int.R_+^2$ of X_1X_2-plane under the following necessary and sufficient conditions:

$$\alpha_1 > (\alpha_4 + \alpha_2) \qquad (2.2)$$

4. disease free equilibrium point $E_3 = (\overline{X}_1, 0, \overline{X}_3)$,

where

$$\overline{X}_1 = \frac{\alpha_6}{e_1\alpha_3(1-n)}, \overline{X}_3 = \frac{e_1\alpha_3(1-n)-\alpha_6}{e_1\alpha_3^2(1-n)^2} \qquad (2.3)$$

exists uniquely $Int.R_+^2$ of X_1X_3-plane under the following necessary and sufficient conditions:

$$\alpha_6 < e_1\alpha_3(1-n) \qquad (2.4)$$

5. The positive equilibrium point $E^* = (X_1{}^*, X_2{}^*, X_3{}^*)$, where

$$X_2{}^* = \frac{\alpha_6 - e_1\alpha_3(1-n)X_1{}^*}{e_2\alpha_5(1-m)}$$

(2.5a)

$$X_3{}^* = \frac{\alpha_1X_1{}^* - \alpha_4 - \alpha_2}{\alpha_5(1-m)}$$

(2.5b)

While $X_1{}^*$ represents a positive root to the following second order equation.

$$B_1X_1^2 + B_2X_1 + B_3 = 0$$

Where

$$B_1 = [e_2(\alpha_5(1-m)+\alpha_1\alpha_3(1-n)) - e_1\alpha_3(1-n)(1+\alpha_1)]$$

$$B_2 = -e_2(\alpha_5(1-m)-\alpha_3(\alpha_2+\alpha_4)(1-n)) + (1+\alpha_1)\alpha_6 + e_1\alpha_2\alpha_3(1-n)$$

$$B_3 = \alpha_2\alpha_6$$

Clearly the last equation has a positive root, namely X_1^*, provided that

$$e_2[\alpha_5(1-m)+\alpha_1\alpha_3(1-n)] < e_1\alpha_3(1-n)(1+\alpha_1) \qquad (2.5c)$$

Therefore E^* exists uniquely in $Int.R_+^3$, if in addition to condition (2.5c) the following condition holds

$$\frac{\alpha_4+\alpha_2}{\alpha_1} < X_1^* < \frac{\alpha_6}{e_1\alpha_3(1-n)} \qquad (2.5d)$$

The local Stability Analysis:

In this section, the local stability analysis of system (1.2) around each of the above equilibrium points are discussed using the linearization technique. Note that from now onward the symbols λ_1^i, λ_2^i and λ_3^i for $i = 0,1,...,4$ are used to represent the eigenvalues of the Jacobian matrix $J(E_i)$ that describe the dynamics in X_1, X_2, X_3-direction respectively.

The Jacobian matrix of system (1.2) at E_0 can be written as:

$$J_0 = J(E_0) = \begin{bmatrix} 1 & \alpha_2 & 0 \\ 0 & -(\alpha_2+\alpha_4) & 0 \\ 0 & 0 & -\alpha_6 \end{bmatrix} \qquad (3.1)$$

Then the eigenvalues of J_0 can be written as

$$\lambda_1^0 = 1 > 0,\ \lambda_2^0 = -(\alpha_2 + \alpha_4) < 0 \ \text{ and } \ \lambda_3^0 = -\alpha_6 < 0$$

Therefore E_0 is a saddle point.

The Jacobian matrix of system (1.2) at E_1 can be written as:

$$J_1 = J(E_1) = \begin{bmatrix} -1 & -1+\alpha_2-\alpha_1 & -\alpha_3(1-n) \\ 0 & -\alpha_1-\alpha_4-\alpha_2 & 0 \\ 0 & 0 & -\alpha_6+e_1\alpha_3(1-n) \end{bmatrix} \tag{3.2a}$$

Thus its eigenvalues can be written as

$$\lambda_1^1 = -1 < 0,\ \lambda_2^1 = -(\alpha_1 + \alpha_2 + \alpha_4) < 0 \ \text{ and } \ \lambda_3^1 = -\alpha_6 + e_1\alpha_3(1-n)$$

Hence E_1 is locally asymptotically stable if the following condition holds.

$$\alpha_6 > e_1\alpha_3(1-n) \tag{3.2b}$$

Otherwise its a saddle point.

The Jacobian matrix of system (1.2) at E_2 can be written as:

$$J_2 = J(E_2) = (\beta_{ij})_{3\times 3} \tag{3.3}$$

where

$\beta_{11} = 1-2\hat{X}_1 - \hat{X}_2(1+\alpha_1); \beta_{12} = -\hat{X}_1(1+\alpha_1)+\alpha_2;\ \beta_{13} = -\alpha_3(1-n)\hat{X}_1;$

$\beta_{21} = \alpha_1\hat{X}_2; \beta_{22} = 0;\ \beta_{23} = -\alpha_5(1-m)\hat{X}_2;$

$\beta_{31} = 0; \beta_{32} = 0;\ \beta_{33} = -\alpha_6 + e_1\alpha_3(1-n)\hat{X}_1 + e_2\alpha_5(1-m)\hat{X}_2$

Then the characteristic equation of J_2 can be written as

$$(\lambda^2 - \beta_{11}\lambda - \beta_{12}\beta_{21})(\beta_{33} - \lambda) = 0$$

(3.4)

So, the eigenvalues of J_2 can be written as

$$\lambda_1^2 = \frac{\beta_{11}}{2} + \frac{1}{2}\sqrt{\beta_{11}{}^2 + 4\beta_{12}\beta_{21}},\ \lambda_2^2 = \frac{\beta_{11}}{2} - \frac{1}{2}\sqrt{\beta_{11}{}^2 + 4\beta_{12}\beta_{21}},\ \lambda_3^2 = \beta_{33}$$

Straightforward computation shows that these eigenvalues have negative real parts provided that the following conditions hold.

$$2\hat{X}_1 - \hat{X}_2(1+\alpha_1) > 1 \tag{3.5a}$$

$$\hat{X}_1(1+\alpha_1) > \alpha_2 \tag{3.5b}$$

$$e_1\alpha_3(1-n)\hat{X}_1 + e_2\alpha_5(1-m)\hat{X}_2 < \alpha_6 \tag{3.5c}$$

The Jacobian matrix of system (1.2) at E_3 can be written as:

$$J_3 = J(E_3) = (\beta_{ij})_{3\times 3} \tag{3.6}$$

Where

$\beta_{11} = 1-2\overline{X}_1 - \alpha_3(1-n)\overline{X}_3; \beta_{12} = -\overline{X}_1(1+\alpha_1)+\alpha_2;\ \beta_{13} = -\alpha_3(1-n)\overline{X}_1$

$\beta_{21} = 0;\ \beta_{22} = \alpha_1\overline{X}_1 - \alpha_4 - \alpha_2 - \alpha_5(1-m)\overline{X}_3; \beta_{23} = 0$

$\beta_{31} = e_1\alpha_3(1-n)\overline{X}_3; \beta_{32} = e_2\alpha_5(1-m)\overline{X}_3; \beta_{33} = 0$

Then the characteristic equation of J_3 can be written as

$$(\beta_{22}-\lambda)(\lambda^2-\beta_{11}\lambda-\beta_{13}\beta_{31})=0 \tag{3.7}$$

So, the eigenvalues of J_3 can be written as

$$\lambda_1^3=\frac{\beta_{11}}{2}+\frac{1}{2}\sqrt{\beta_{11}{}^2+4\beta_{13}\beta_{31}},\ \lambda_2^3=\beta_{22},\ \lambda_3^3=\frac{\beta_{11}}{2}-\frac{1}{2}\sqrt{\beta_{11}{}^2+4\beta_{13}\beta_{31}}$$

Again straightforward computation shows that these eigenvalues have negative real parts provided that the following conditions hold.

$$1<2\overline{X}_1+\alpha_3(1-n)\overline{X}_3 \tag{3.8a}$$

$$\alpha_1\overline{X}_1<\alpha_4+\alpha_2+\alpha_5(1-m)\overline{X}_3 \tag{3.8b}$$

The Jacobian matrix of system (1.2) at E^* can be written as:

$$J^*=J(E^*)=(\beta_{ij})_{3\times 3} \tag{3.9}$$

where

$$\beta_{11}=1-2X_1^*-X_2^*-\alpha_1X_2^*-\alpha_3(1-n)X_3^*$$
$$\beta_{12}=-X_1^*(1+\alpha_1)+\alpha_2;\beta_{13}=-\alpha_3(1-n)X_1^*$$
$$\beta_{21}=\alpha_1X_2^*;\beta_{22}=0;\beta_{23}=-\alpha_5(1-m)X_2^*$$
$$\beta_{31}=e_1\alpha_3(1-n)X_3^*;\beta_{32}=e_2\alpha_5(1-m)X_3^*;\beta_{33}=0$$

Therefore, the locally stability conditions are established in the following theorem.

Theorem (1.2): The positive equilibrium point E^* of the system (1.2) is locally asymptotically stable in $Int.R_+^3$, if and only if the following conditions are satisfied:

$$[2X_1^*+X_2^*(1+\alpha_1)+\alpha_3(1-n)X_3^*]>1 \tag{3.10a}$$

$$[X_1^*(1+\alpha_1)]>\alpha_2 \tag{3.10b}$$

$$X_2^*<\frac{e_1\alpha_3(1-n)[1-2X_1^*-\alpha_3(1-n)X_3^*}{e_2\alpha_1\alpha_5(1-m)+e_1\alpha_3(1-n)(1+\alpha_1)} \tag{3.10c}$$

Proof: [ ]

Persistence of system (1.2)

In this section, the persistence of system (1.2) is studied. It is well known that the system is said to be persistence if and only if each species persists. Mathematically this is meaning that the solution of system (1.2) do not have omega limit set in the boundaries of R_+^3. Therefore, in the following theorem, the necessary and sufficient conditions for the uniform persistence of the system (1.2) are derived.

***Theorem (4.1)*:** Assume that there are no periodic dynamics in the boundary planes X_1X_2, X_1X_3 *and* X_2X_3 respectively. Further, if the following conditions are hold.

$$\gamma_2(\alpha_1-\alpha_4-\alpha_2)>\gamma_3(\alpha_6-e_1\alpha_3(1-n)) \tag{4.1a}$$

$$(\hat{X}_1 + \hat{X}_2(1+\alpha_1)) < \left[1 + \frac{\alpha_2 \hat{X}_2}{\hat{X}_1}\right] \quad (4.1b)$$

$$1 > (\overline{X}_1 + \alpha_3(1-n)\overline{X}_3) \quad (4.1c)$$

Then, system (1.2) is uniformly persistence.

Proof. You can see [10]

Globally analysis

In the following the global dynamics of system (1.2) is carried out as shown in the following theorems.

Theorem (5.1): Assume that the axial equilibrium point $E_1 = (1,0,0)$ of system (1.2) is locally asymptotically stable in R_+^3 and let that

$$\tilde{X}_1 < \min\left[\frac{\alpha_4}{1+\alpha_1}, \frac{\alpha_6}{\alpha_3(1-n)}\right] \quad (5.1)$$

Then E_1 is globally asymptotically stable in R_+^3.

Proof: you can see [10]

Theorem (5.2): Assume that E_2 is locally asymptotically stable point in R_+^3 then E_2 is globally asymptotically stable on the sub region of R_+^3 that satisfies the following conditions:

$$[e_1\alpha_3(1-n)\hat{X}_1 + e_2\alpha_5(1-m)\hat{X}_2] < \alpha_6 \quad (5.2a)$$

$$\frac{\alpha_2}{1+\alpha_1} < X_1 \quad (5.2b)$$

$$e_2\alpha_1 < e_1\left(1+\alpha_1 - \frac{\alpha_2}{X_1}\right) \quad (5.2c)$$

$$M_3 < M_1 + M_2 \quad (5.2d)$$

Where

M_1, M_2 and M_3 are given in the proof.

Proof: you can see [10]

Theorem (5.3): Assume that the equilibrium point E_3 of system (1.2) is locally asymptotically stable in R_+^3 with the following conditions:

$$\frac{\alpha_2}{1+\alpha_1} < X_1 \quad (5.3a)$$

$$\frac{e_2}{e_1}\left(\alpha_5(1-m)\overline{X}_3 + \alpha_2 + \alpha_4\right) > \frac{e_2}{e_1}\alpha_1 X_1 + \left(1+\alpha_1 - \frac{\alpha_2}{X_1}\right)\overline{X}_1 \quad (5.3b)$$

Then E_3 is globally asymptotically stable in the sub region of R_+^3 that satisfy the above conditions.

Proof: you can see [10]

Theorem (5.4): Assume that E^* is a locally asymptotically stable point in Int. R_+^3 then E^* is globally asymptotically stable in the sub region of $Int.R_+^3$ that satisfies the following conditions:

$$e_1(1+\alpha_1) > \frac{e_1\alpha_2}{X_1} - e_2\alpha_1 \tag{5.4a}$$

$$N_1 > N_2 \tag{5.4b}$$

where

N_1 and N_2 are given in the proof.

Proof: you can see [10]

Numerical analysis:

In this section the global dynamics of system (1.2) is studied numerically. The objectives of this study are confirming our analytical results and understand the effects of the parameters including the refuge rate on the dynamics of SIS epidemic system. Consequently, system (1.2) is solved numerically, for different sets of parameters and for different sets of initial conditions. It is observed that, for the following set of parameters, system (1.2) is solved numerically at different sets of initial values and then the trajectories of system (1.2) are drawn in Fig. (6.1).

$$\begin{aligned}&\alpha_1 = 0.5, \alpha_2 = 0.05, \alpha_3 = 0.3, \alpha_4 = 0.05, \alpha_5 = 0.4,\\ &\alpha_6 = 0.1, e_1 = 0.8,\ e_2 = 0.9, m = 0.5, n = 0.75\end{aligned} \tag{6.1}$$

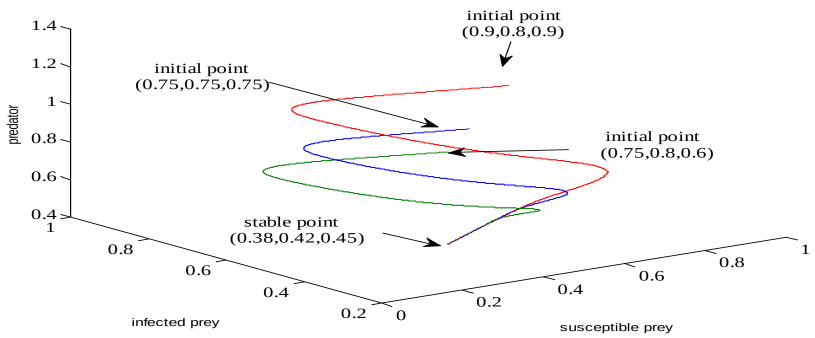


Figure (6.1): Phase plot of system (1.2) for the data given by Eq. (6.1) starting from different initial points..

In the above figure, system (1.2) approaches asymptotically to the stable coexistence equilibrium point $E^* = (0.38, 0.42, 0.45)$, starting from different initial points. Clearly, Fig. (6.1) shows the existence of a unique endemic equilibrium point of system (1.2) which is globally asymptotically stable.

Note that for the time series figures we will use throughout this section that the solid line for describing the trajectory of X_1 ; dotted line for describing the trajectory of X_2 ; dashed line for describing the trajectory of X_3.

Now, in order to discuss the effect of infection rate α_1 on the dynamical behavior of system (1.2). The system is solved numerically for different values of α_1 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.2a)-(6.2c) for the typical values $\alpha_1 = 0.05, 0.15, 0.9$.

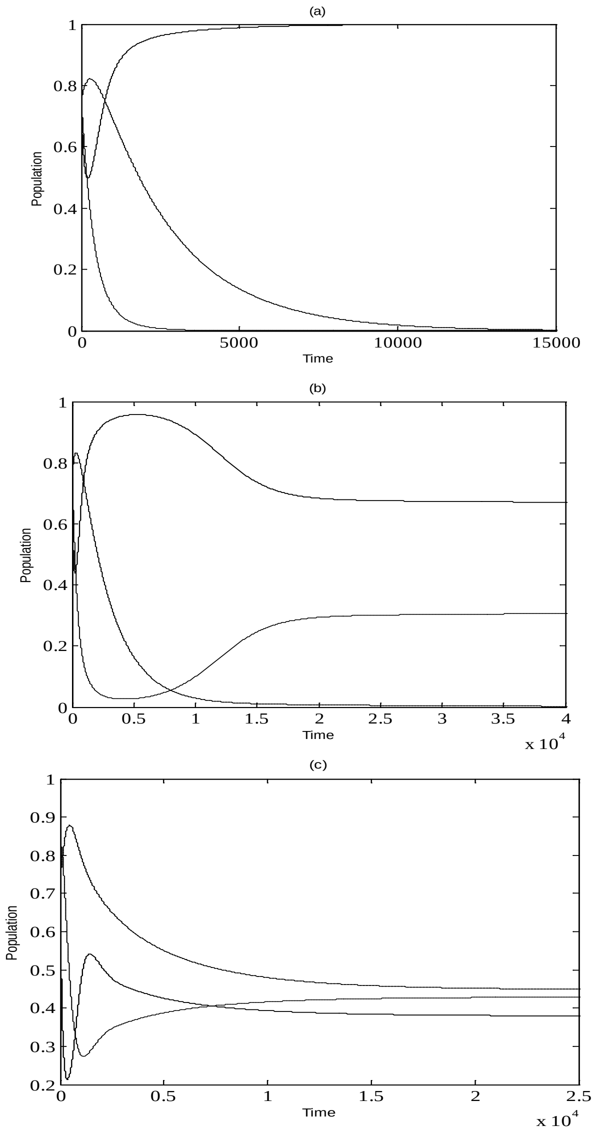


Figure. (6.2*): Time series of system (1.2) for the data given by Eq. (6.1) with: (a) for* $\alpha_1 = 0.05$*. (b) for* $\alpha_1 = 0.15$*. (c) for* $\alpha_1 = 0.9$.

According to the above figure, it is clear that, Fig.(6.2a) shows the approaching of system (1.2) to $E_1 = (1,0,0)$, and Fig.(6.2b) shows the approaching of system (1.2) to free predator equilibrium point $E_2 = (0.66,0.30,0)$. However, Fig.(6.2c) shows the approaching of system (1.2) to the positive equilibrium point $E^* = (0.17,0.49,0.29)$. Moreover, its observed that for $\alpha_1 \leq 0.1$ the solution approaches asymptotically to

$E_1 = (1,0,0)$, however increasing the value of infection rate in the range $0.1 < \alpha_1 \le 0.16$ the solution approaches asymptotically to $E_2 = (\hat{X}_1, \hat{X}_2, 0)$. Finally, for $\alpha_1 \ge 0.17$, E_2 becomes unstable and the solution approaches asymptotically to $E^* = (X_1^*, X_2^*, X_3^*)$.

The effect of the recover rate α_2 on the dynamical behavior of system (1.2) is studied through solving the system numerically for different values of α_2 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.3a)-(6.3c) for the typical values $\alpha_2 = 0.01, 0.35, 0.5$.

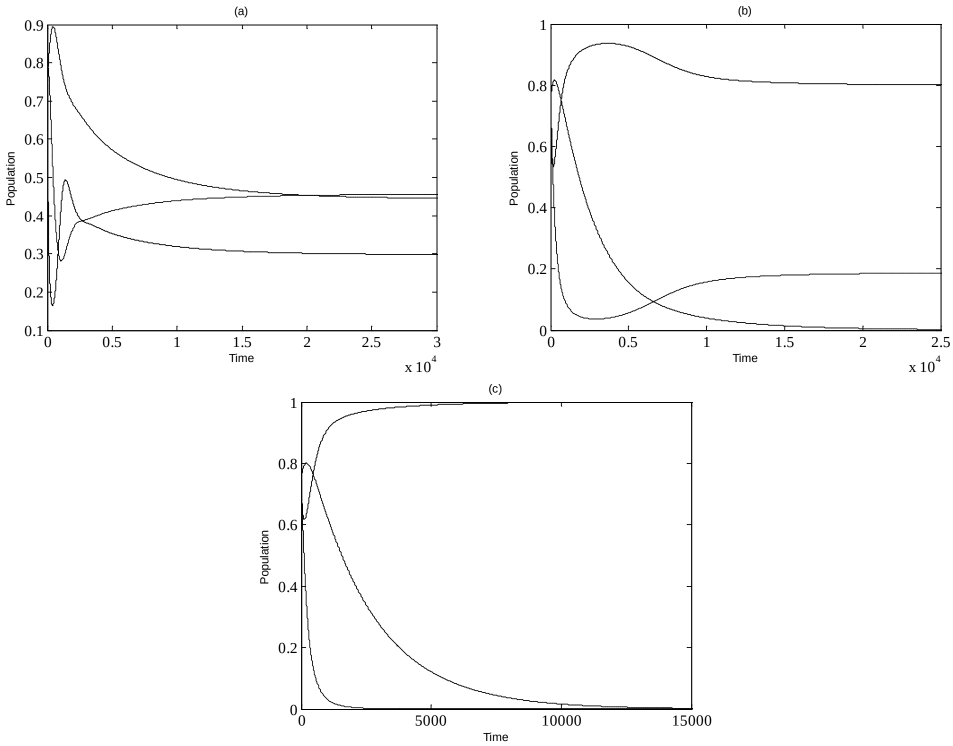


Figure.(6.3): Time series of solution of system (1.2). (a) for $\alpha_2 = 0.01$, (b) for $\alpha_2 = 0.35$, (c) for $\alpha_2 = 0.5$.

According to the above figure, it is clear that, Fig.(6.3a) shows the approaching of system (1.2) to the positive equilibrium point $E^* = (0.29, 0.45, 0.44)$, and Fig.(6.3b) shows the approaching of system (1.2) to free predator equilibrium point $E_2 = (0.8, 0.188, 0)$. However, Fig.(6.3c) shows the approaching of system (1.2) to $E_1 = (1,0,0)$. Moreover, its observed that for $\alpha_2 \le 0.19$ the solution approaches asymptotically to $E^* = (X_1^*, X_2^*, X_3^*)$, however increasing the value of recover rate in the range $0.19 < \alpha_2 \le 0.44$ the solution approaches asymptotically to $E_2 = (\hat{X}_1, \hat{X}_2, 0)$. Finally, for $\alpha_2 \ge 0.45$, E_2 becomes unstable and the solution approaches asymptotically to $E_1 = (1,0,0)$.

The effect of predation rate α_3 on the dynamical behavior of system (1.2) is discussed by solving system (1.2) numerically for different values of α_3 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.4a)-(6.4b) for the typical values $\alpha_3 = 0.1, 0.8$.

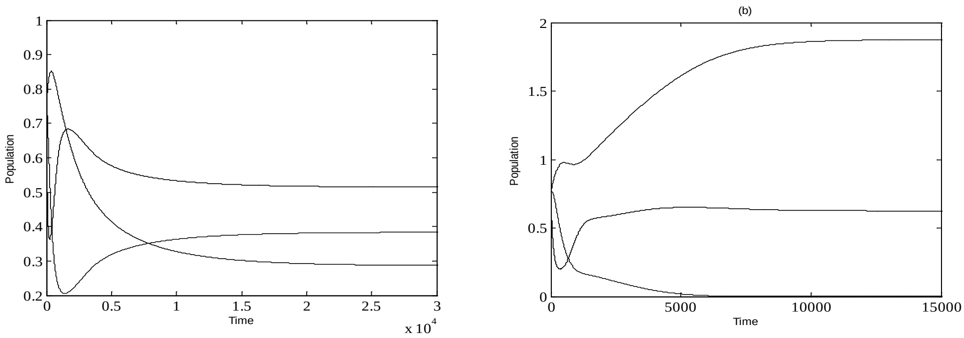
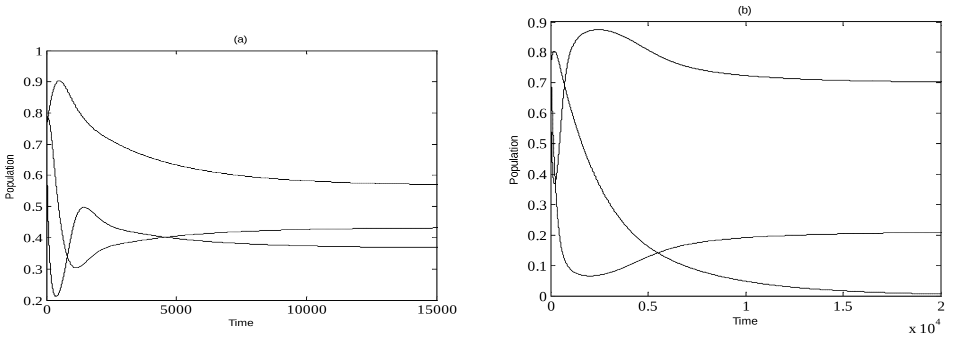


Figure.(6.4): Time series of the solution of the system (1.2). (a) for $\alpha_3 = 0.1$,(b) for $\alpha_3 = 0.8$.

From Fig.(6.4a) it is observed that system (1.2) approaches to the positive equilibrium point $E^* = (0.29, 0.52, 0.24)$, while Fig.(6.4b) shows the approaching of system to the disease free equilibrium point $E_3 = (0.62, 0, 1.87)$. Moreover, its observed that for $\alpha_3 \le 0.64$ the solution approaches asymptotically to $E^* = (X_1^*, X_2^*, X_3^*)$, and, for $\alpha_3 \ge 0.65$, E^* becomes unstable and the solution approaches asymptotically to $E_3 = (\overline{X}_1, 0, \overline{X}_3)$.

The effect of the disease death rate α_4 on the dynamical behavior of system (1.2) is also studied through solving the system (1.2) numerically for different values of α_4 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.5a)-(6.5c) for the typical values $\alpha_4 = 0.02, 0.3, 0.45$.



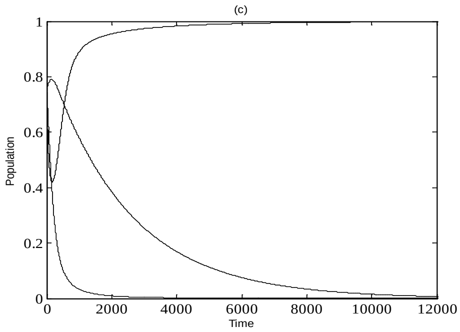


Figure.(6.5): Time series of solution of system (1.2). (a) for $\alpha_4 = 0.02$, (b) for $\alpha_4 = 0.3$, (c) for $\alpha_4 = 0.45$.

According to the above figure, it is clear that, Fig.(6.5a) shows the approaching of system (1.2) to the positive equilibrium point $E^* = (0.36, 0.43, 0.56)$, while Fig.(6.5b) shows the approaching of system (1.2) to the predator free equilibrium point $E_2 = (0.7, 0.21, 0)$. However, Fig.(6.5c) shows the approaching of system (1.2) to $E_1 = (1,0,0)$. Moreover, its observed that for $\alpha_4 \leq 0.16$ the solution approaches asymptotically to $E^* = (X_1^*, X_2^*, X_3^*)$, however increasing the value of disease death rate in the range $0.17 \leq \alpha_4 \leq 0.44$ causes extinction in predator species and the solution approaches asymptotically to predator free equilibrium point $E_2 = (\hat{X}_1, \hat{X}_2, 0)$. Finally, for $\alpha_4 \geq 0.45$, E_2 becomes unstable and the solution approaches asymptotically to $E_1(1,0,0)$.

Now the effect of the predation rate α_5 on the dynamical behavior of system (1.2) is investigated numerically by solving the system (1.2) for different values of α_5 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.6a)-(6.6b) for the typical values $\alpha_5 =$ (0.1,0.9).

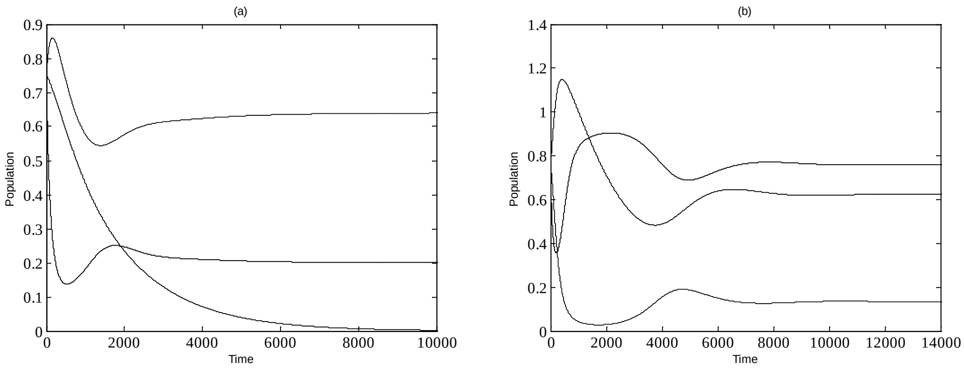


Figure.(6.6): Time series of the solution of system (1.2). (a) for $\alpha_5 = 0.1$ (b) for $\alpha_5 = 0.9$.

Clearly Fig.(6.6a) shows the approaching of system (1.2) to the predator free equilibrium point $E_2 = (0.2, 0.64, 0)$ however Fig.(6.6b) shows the approaching of system (1.2) to the positive equilibrium point $E^{*}=(0.76, 0.13, 0.62)$. Moreover, its observed that for $\alpha_5 \leq 0.27$ the solution approaches asymptotically to predator free equilibrium point $E_2=(\hat{X}_1, \hat{X}_2, 0)$, and, for $\alpha_5 \geq 0.28$, E_2 becomes unstable and the solution approaches asymptotically to $E^{*}=(X_1^{*}, X_2^{*}, X_3^{*})$.

The effect of the predator death rate α_6 on the dynamical behavior of system (1.2) is discussed by solving system (1.2) numerically for different values of α_6 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.7a)-(6.7c) for the typical values $\alpha_6 = 0.05, 0.07, 0.9$.

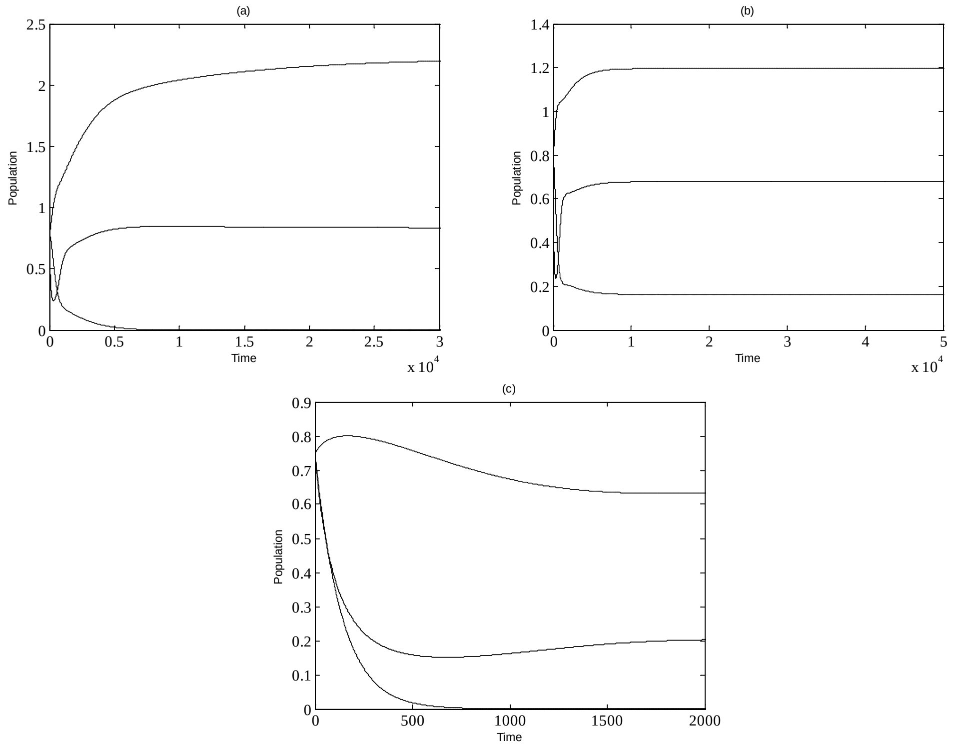


Figure.(6.7): Time series of the solution of system (1.2). (a) for $\alpha_6 = 0.05$ (b) for $\alpha_6 = 0.07$ (c) for $\alpha_6 = 0.9$

According to the above figure, it is clear that, Fig.(6.7a) shows the approaching of system (1.2) to the disease free equilibrium point $E_3=(0.83, 0, 2.21)$, and Fig.(6.7b) shows the approaching of system (1.2) to the positive equilibrium point $E^{*}=(0.67, 0.16, 1.19)$, while Fig.(6.7c) shows the approaching of system (1.2) to the predator free equilibrium point $E_2=(0.2, 0.64, 0)$. Moreover, its observed that for $\alpha_6 < 0.05$ the solution approaches asymptotically to the disease free equilibrium point $E_3=(\overline{X}_1, 0, \overline{X}_3)$, however increasing the value of predation death rate in the range $0.05 \leq \alpha_6 \leq 0.12$ causes destabilization in $E_3=(\overline{X}_1, 0, \overline{X}_3)$ and the solution approaches asymptotically to positive equilibrium point $E^{*}=(X_1^{*},X_2^{*},X_3^{*})$. Finally, for $\alpha_6 \geq 0.13$, E^{*} becomes unstable and the solution approaches asymptotically to the predator free equilibrium point $E_2=(\hat{X}_1,\hat{X}_2,0)$.

Further, it is observed that varying the conversion rate e_1 keeping other parameters as given in Eq. (6.1) does not has any effect on the dynamical behavior of system (1.2) and the system (1.2) still approaches asymptotically to the positive equilibrium point $E^{*}=(X_1^{*},X_2^{*},X_3^{*})$.

Now the effect of the conversion rate e_2 on the dynamical behavior of system (1.2) is discussed through solving the system (1.2) numerically for different values of e_2 keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.8a)-(6.8b) for the typical values $e_2=0.5, 1$.

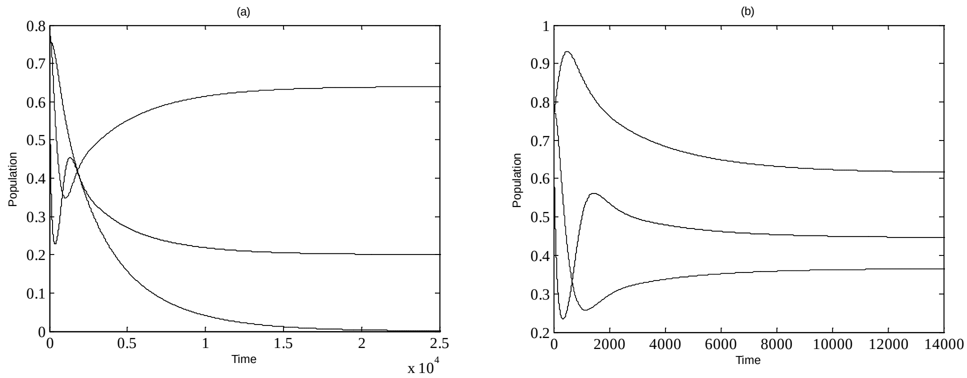


Figure.(6.8): Time series of the solution of system (1.2). (a) for $e_2=0.5$. (b) for $e_2=1$.

From Fig.(6.8a) we see that the system approaching to predator free equilibrium point $E_2=(0.2, 0.64, 0)$, while Fig.(6.8b) shows the approaching of system (1.2) to the positive equilibrium point $E^{*}=(0.44, 0.36, 0.61)$. Moreover, its observed that for $e_2 \leq 0.66$ the solution of system (1.2) approaches asymptotically to predator free equilibrium point $E_2=(\hat{X}_1,\hat{X}_2,0)$, however for $e_2 \geq 0.67$, E_2 becomes unstable and the solution of system (1.2) approaches asymptotically to $E^{*}=(X_1^{*},X_2^{*},X_3^{*})$.

The effect of the susceptible refuge protection rate *n* on the dynamical behavior of system (1.2) is studied by solving system (1.2) numerically for different values of *n* keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.9a)-(6.9b) for the typical values $n=0.1, 0.9$.

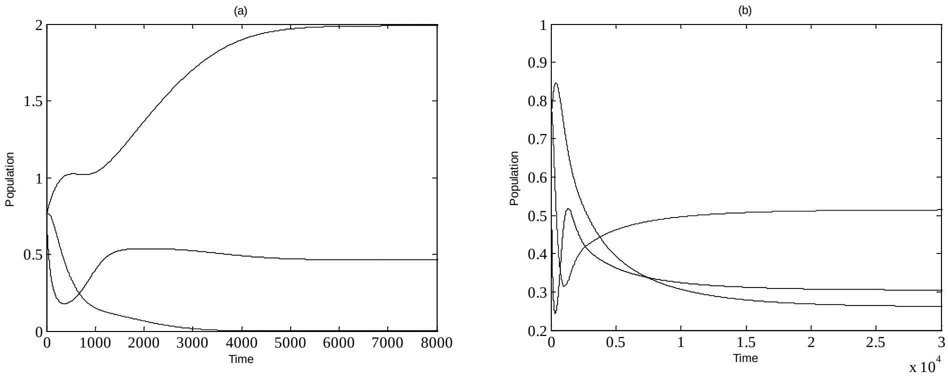


Figure.(6.9): Time series of the solution of system (1.2). (a) for $n = 0.1$ (b) for $n = 0.9$.

According to the Fig.(6.9a) the system (1.2) approaches asymptotically to the disease free equilibrium point $E_3 = (0.46, 0, 1.98)$, while Fig.(6.9b) shows the approaching of system (1.2) to the positive equilibrium point $E^* = (0.3, 0.51, 0.26)$. Moreover, its observed that for $n \le 0.46$ the solution of system (1.2) approaches asymptotically to the disease free equilibrium point $E_3 = (\overline{X}_1, 0, \overline{X}_3)$, however for $n \ge 0.47$, E_3 becomes unstable and the solution approaches asymptotically to $E^* = (X_1^*, X_2^*, X_3^*)$.

The effect of the infected refuge protection rate *m* on the dynamical behavior of system (1.2) is also studied by solving the system (1.2) numerically for different values of *m* keeping other parameters fixed as given in Eq. (6.1), and then the solution of system (1.2) as a function of time is drawn in Fig. (6.10a)-(6.10b) for the typical values $m = 0.1, 0.7$.

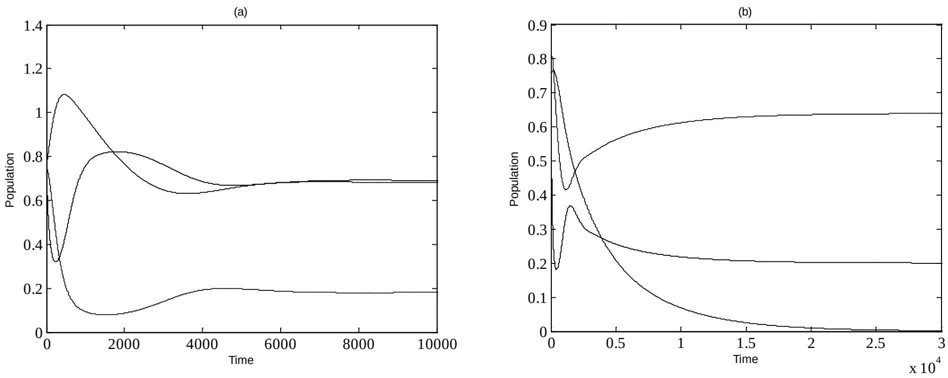


Figure.(6.10): Time series of the solution of system (1.2). (a) for $m = 0.1$ (b) for $m = 0.7$.

From the Fig.(6.10a) it is clear that system (1.2) approaching asymptotically to the positive equilibrium point $E^* = (0.69, 0.18, 0.68)$, while Fig.(6.10b) shows the approaching of system to the predator free equilibrium point $E_2 = (0.2, 0.64, 0)$. Moreover, its observed that for $m \le 0.63$ the solution of system (1.2) approaches asymptotically to $E^{*} = (X_1^{*}, X_2^{*}, X_3^{*})$, however for $m \geq 0.64$, E^{*} becomes unstable and the solution approaches asymptotically to $E_2 = (\hat{X}_1, \hat{X}_2, 0)$.

Discussion and conclusion:

In this chapter, a prey-predator model involving prey refuge with Lotka-Volttera functional response is proposed and analyzed. The stability analysis (local and global) of the equilibrium pointes of the proposed system is carried out. The boundedness and permanence of the system have been proved. In order to study the effect of system parameters involving the refuge on the dynamical behavior of the system, a numerical work has been done taking into account the set values of the parameters in (6.1) and the results can be summarized as follow:

1. Decreasing the infection rate α_1, in the range $0.1 < \alpha_1 \leq 0.16$, causes extinction in predator species first and then decreasing the infection rate further, in the range $\alpha_1 \leq 0.1$, leads to extinction in an infected prey species.
2. Increasing the recover rate α_2, in the range $0.19 < \alpha_2 \leq 0.44$, causes extinction in predator species first and then increasing the recover rate further $\alpha_2 \geq 0.45$ leads to extinction in an infected prey species.
3. Increasing the predation rate α_3, in the range $\alpha_3 \geq 0.65$, causes extinction in an infected prey species.
4. The disease death rate has the same effects as that of the recover rate α_2 on the dynamical behavior of system (1.2).
5. Decreasing the infection rate α_5, in the range $\alpha_5 \leq 0.27$, causes extinction in predator species
6. Decreasing the predation death rate α_6, in the range $\alpha_6 < 0.05$, causes extinction in an infected prey species. However, increasing the predation death rate α_6, in the range $\alpha_6 \geq 0.13$, causes extinction in predator species.
7. The conversion rate e_2 has the same effects as that of the infection rate α_5 on the dynamical behavior of system (1.2). On the other hand the conversion rate e_1 dose not has any effect on the dynamics of sysytem (1.2).
8. Decreasing the susceptible refuge rate *n*, in the range $n \leq 0.46$, causes extinction in an infected prey species.
9. Increasing the infected refuge rate *m*, in the range $m \geq 0.64$, causes extinction in an predator species.

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