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Classification of Critical Points and Caustic of the Functions of Codimensions Eight and Fifteen

تصنيف النقاط الحرجة وكاوستيك للدوال التي أبعادها المرافقة ثمانية و خمسة عشر

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Abstract.
In the present paper we classified the critical points of functions of codimensions eight and fifteen by finding the geometric description of caustic, investigate the propagation of the critical points and determine the level curves of these functions in the space of parameters.

الخلاصة
في البحث الحالي قمنا بتصنيف النقاط الحرجة للدوال التي تكون أبعادها المرافقة ثمانية و خمسة عشر وذلك من خلال إيجاد الوصف الهندسي للكاوستيك وكذلك إيجاد أنواع النقاط الحرجة ومنحنيات المستوى لهذة الدوال في فضاء المعلمات.

1. Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

$$F(x,\lambda,) = b, \quad x \in O \subset X, \quad b \in Y, \quad \lambda \in R^n. \qquad (1.1)$$

where X, Y are real Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\Theta(\xi,\lambda) = \beta, \quad \xi \in \hat{M}, \quad \beta \in \hat{N}. \qquad (1.2)$$

can be used, where $\widehat{M}$ and $\widehat{N}$ are smooth finite dimensional manifolds. The method used for these studies is the Lyapunov-Shmidt method [9,11].

Definition 1.1 [9] Suppose that E and F are Banach spaces and $A: E \to F$ be a linear continuous operator. The operator A is called Fredholm operator, if

1- The kernel of A, Ker(A), is finite dimensional,
2- The rang of A, Im(A), is closed in F,
3- The Cokernel of A, Coker(A), is finite dimensional.

The number

$dim(Ker\ A) - dim(Coker\ A)$

is called Fredholm index of the operator A.

Suppose that $f: \Omega \to F$ is a nonlinear Fredholm map of index zero. A smooth map $f: \Omega \to F$ has variational property, if there exist a functional $V: \Omega \to R$ such that $f = grad_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H, \quad \forall\ x \in \Omega,\ h \in E.$$

where ($\langle \cdot,\cdot \rangle_H$ is the scalar product in Hilbert space H). In this case the solutions of equation $f(x,\lambda) = 0$ are the critical points of functional $V(x,\lambda)$. Suppose that $f: E \to F$ is a smooth Fredholm map of index zero, E, F are Banach spaces and

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda),h\rangle_H, \quad h \in E.$$

where V is a smooth functional on E. Also we assume that $E \subset F \subset H$, H is a Hilbert space, then by using method of finite dimensional reduction (Local scheme of Lyapunov-Schmidt) the problem,

$$V(x,\lambda) \to extr, \quad x \in E, \ \lambda \in R^n.$$

can be reduced into equivalent problem,

$$W(\xi,\lambda) \to extr, \quad \xi \in R^n.$$

the function $W(\xi,\lambda)$ is called key function.

If $N = span\{e_1,...,e_n\}$ is a subspace of E, where $e_1,..., e_n$ are orthonormal basis, then the key function $W(\xi,\lambda)$ can be defined in the form,

$$W(\xi,\lambda) = \inf_{x:\langle x,e_i\rangle=\xi_i \ \forall i} V(x,\lambda), \quad \xi = (\xi_1,...,\xi_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, *etc*) [9]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function W. The theory of singularities of smooth functions play an important role in many fields of applications such as planetary systems with satellites, integrable Hamiltonian systems with 2 degrees of freedom, certain BVPs, *etc*. There are many studies about singularities of smooth functions by different authors. The classification of singularities of smooth functions have been studied by Arnold [2,3,4] and other authors. The study of singularities of smooth functions and their applications to BVPs by using Lyapunov-Schmidt reduction have been studied by Sapronov and Darinskii [5]. They studied the bifurcation of the critical points of the following function

$$\tilde{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + 4\eta_1^2\eta_2^2 + \lambda_1\eta_1^2 + \lambda_2\eta_2^2, \qquad \gamma = (\lambda_1,\lambda_2) \in R^2. \tag{1.3}$$

which is appear in the study of a certain variational boundary value problem of fourth-order equation. Also, Abdul Hussain [1] has been studied the bifurcation of the critical points of function (1.3) with four parameters

$$\tilde{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + 4\eta_1^2\eta_2^2 + \lambda_1\eta_1^2 + \lambda_2\eta_2^2 + q_1\eta_1 + q_2\eta_2, \qquad \gamma = (\lambda_1,\lambda_2,q_1,q_2) \in R^4.$$

In [10] Shanan has been found a new geometric description of caustic and investigated the propagation of the critical points of the following function (for more details see [10])

$$\tilde{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + \eta_3^4 + 4(\eta_1^2\eta_2^2 + \eta_1^2\eta_3^2 + \eta_3^2\eta_2^2 + \eta_3\eta_1\eta_2^2 - \frac{1}{3}\eta_1^3\eta_3)$$
$$+ \frac{24\ r_1}{35}\eta_3^2\eta_1 - \frac{8\ r_1}{45}\eta_1^2\eta_3 + \frac{32\ r_1}{45}\eta_2^2\eta_1 + \frac{8\ r_1}{27}\eta_1^3 + \frac{8\ r_1}{81}\eta_3^3 + \frac{32\ r_1}{63}\eta_2^2\eta_3$$
$$+ k_1\eta_1^2 + k_2\eta_2^2 + k_3\eta_3^2 - q_1\eta_1 - q_3\eta_3, \qquad \gamma = (k_1,k_2,k_3,q_1,q_3) \in R^5$$

Kadem in [7] has been studied the classifications of the critical points of the following two functions that appears in the study of bifurcation solutions of Duffing equation (for more details see [7])

$$\widehat{W}(x,y,\lambda_1,\lambda_2,\lambda_3) = \frac{x^3}{3} + xy^2 + \lambda_1(x^2 - y^2) + \lambda_2 x + \lambda_3 y$$

$$\widetilde{W}(x_1,x_2,\lambda_1,\lambda_2) = \frac{x_1^5}{5} + x_1x_2^4 + x_1^3x_2^2 + \frac{x_1^3}{3} + x_1x_2^2 + \lambda_1x_1^2 + \lambda_2x_2^2$$

In his study he used the boundary singularities of smooth functions to invistegate the propagation of the critical points of the second function. The behavior of smooth functions near the singular points is useful in the study of bifurcation solutions of certain nonlinear differential equations, so the main problem in this work is to give a geometric description of caustic, investigate the propagation of the critical points and determine the level curves of the certain functions. Two examples we introduced in this paper, the first about the classification of the critical points of the function of codimsension eight and the other was the classification of the critical points of function of codimsension fifteen. All these functions referred to the key functions obtain as the reduced functions of the certain functionals (for more details see [7],[8],[9],[10],[11]).

Definition 1.2 [7] Two maps $f, g : \mathbb{R}^n \to \mathbb{R}^k$ are said to be germ-equivalent at $p \in \mathbb{R}^n$ if p is in the domain of both and there is a neighborhood U of p such that the restrictions coincide: $f|_U = g|_U$. A map-germ or function-germ at a point p is an equivalence class of germ-equivalent maps.

Denote by ϵ_n the set of all germs at the origin of smooth functions on $\mathbb{R}^n$. The ring ϵ_n has an important ideal m_n consisting of all smooth function germs vanishing at the origin : $m_n = \{f \in \epsilon_n\colon f(0) = 0\}$. In fact this is a maximal ideal , and indeed the only maximal ideal in ϵ_n, which makes ϵ_n into a local ring as by definition a local ring is one with a unique maximal ideal .

Definition 1.3 [7] (Local algebra). The local algebra Q_f of the singularity of f at the origin is the quotient of the algebra of function-germs by the gradient ideal of f :
$Q_f = \epsilon_n/\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots , \frac{\partial f}{\partial x_n}\right)$
The *multiplicity* $\mu(f)$ of the critical point is the dimension of its local algebra:
$\mu(f) = \dim Q_f$
A critical point is said to be isolated if $\mu(f) < \infty$.

Theorem 1.1 [6] The multiplicity of an isolated critical point is equal to the number of Morse (Nondegenerate) critical points into which it decomposes under a generic deformation of the function .

In section (2) we will explain theorem (1.1). To avoid the singularities of the function we must find a geometric description of the Caustic of this function, so we gave the following definition.

Definition 1.4 [7] The Caustic of the real valued function $\widetilde{W}(x,\lambda)$, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$ is defined to be the set of all λ for which the function $\widetilde{W}(x,\lambda)$ has degenerate critical points.

2. Classification of the Critical Points for Functions of Codimension Eight.

In this section we shall investigate the propagation of the critical points for functions of codimension eight and then determine the type of these points with the level curves of the functions in the space of parameters.

Let $W_0(x,y) = \frac{x^4}{4} + \frac{y^4}{4}$, then the ideal generated by the partial derivatives of W_0 is given by

$$I_{W_0} = \left(\frac{\partial W_0}{\partial x}, \frac{\partial W_0}{\partial y}\right) = (x^3, y^3)$$

hence, the function W_0 has multiplicity equal to nine and Codimension equal to eight so by theorem (1) the number of Morse critical points of this function in nine. The basis for m_2/I_{W_0} can be taken to be $\{x^2y^2, x^2y, xy^2, xy, x^2, y^2, x, y\}$ and then the versal unfolding of $W_0(x, y)$ is given by

$$W_0(x,y,\lambda) = \frac{x^4}{4} + \frac{y^4}{4} + a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x^2 + a_6y^2 + a_7x + a_8y \qquad \text{.....(2.1)}$$

in particular, we shall study function (2.1) in two cases the first when $a_2 = a_3 = 0,\ a_1 = 1$ and the second when $a_4 = 0$ because these two functions are appear as a reduced functions for some nonlinear differential equations (for more details see [7])

Case I:

When $a_2 = a_3 = 0$ the function (2.1) take the following form

$$W(x,y,\lambda) = \frac{x^4}{4} + \frac{y^4}{4} + x^2y^2 + k_1xy + q_1x^2 + q_2y^2 - k_2x - k_3y \qquad \text{... (2.2)}$$

where $a_4 = k_1, a_5 = q_1, a_6 = q_2, a_7 = -k_2$ and $a_8 = -k_3$

Our goal is to find the geometric description of the Caustic of the function (2.2) and then classify the critical points of this function by determine the type of the critical points and the level curves. The critical points of the function (2.2) are the solutions of the following system of nonlinear algebraic equations,

$$x^3 + 2xy^2 + k_1y + 2q_1x - k_2 = 0$$
$$y^3 + 2x^2y + k_1x + 2q_2y - k_3 = 0.$$

all the critical points of function (2.2) are degenerate on the surface given by the equation,

$$6x^4 + 6y^4 - 3x^2y^2 + (6q_2 + 4q_1)x^2 + (6q_1 + 4q_2)y^2 - 8k_1xy - k_1^2 + 4q_1q_2 = 0 \qquad (2.3)$$

To gate the caustic of the function (2.2) we make the following parameterization,

$$k_2 = x^3 + 2xy^2 + k_1y + 2q_1x$$
$$k_3 = y^3 + 2x^2y + k_1x + 2q_2y,$$

By changing the values of k_1, q_1 and q_2 we have the following geometric descriptions of the caustic of function (2.2) in the k_2k_3 −plane, all figures have been drown by using Maple13.

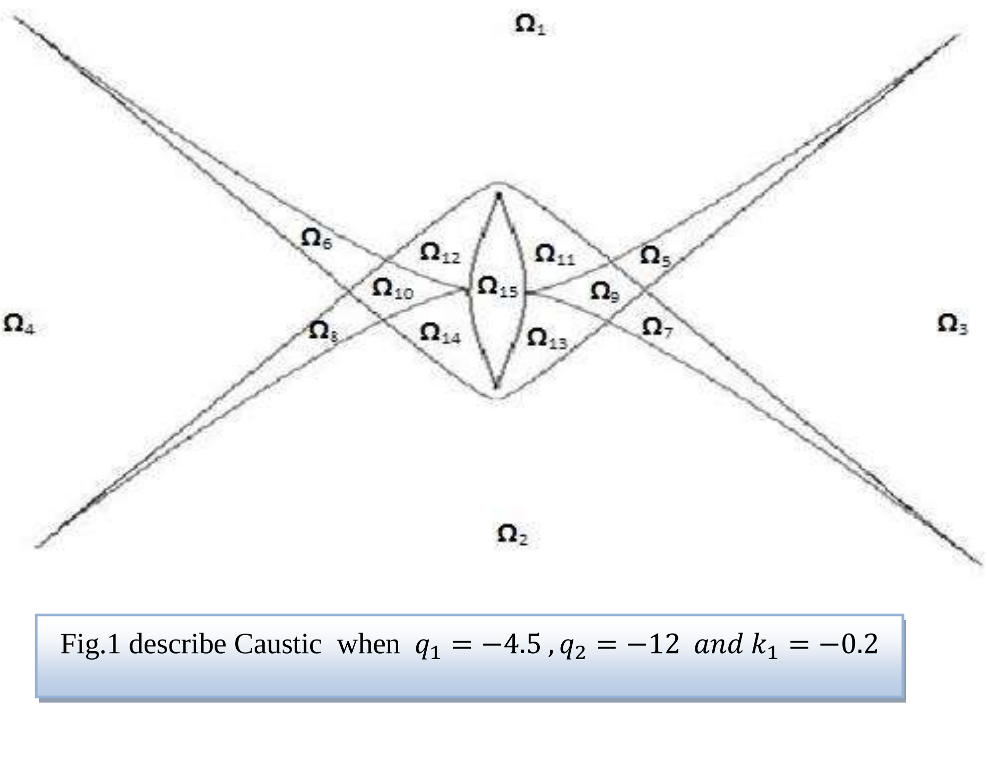


Fig.1 describe Caustic when $q_1 = -4.5\,, q_2 = -12$ *and* $k_1 = -0.2$

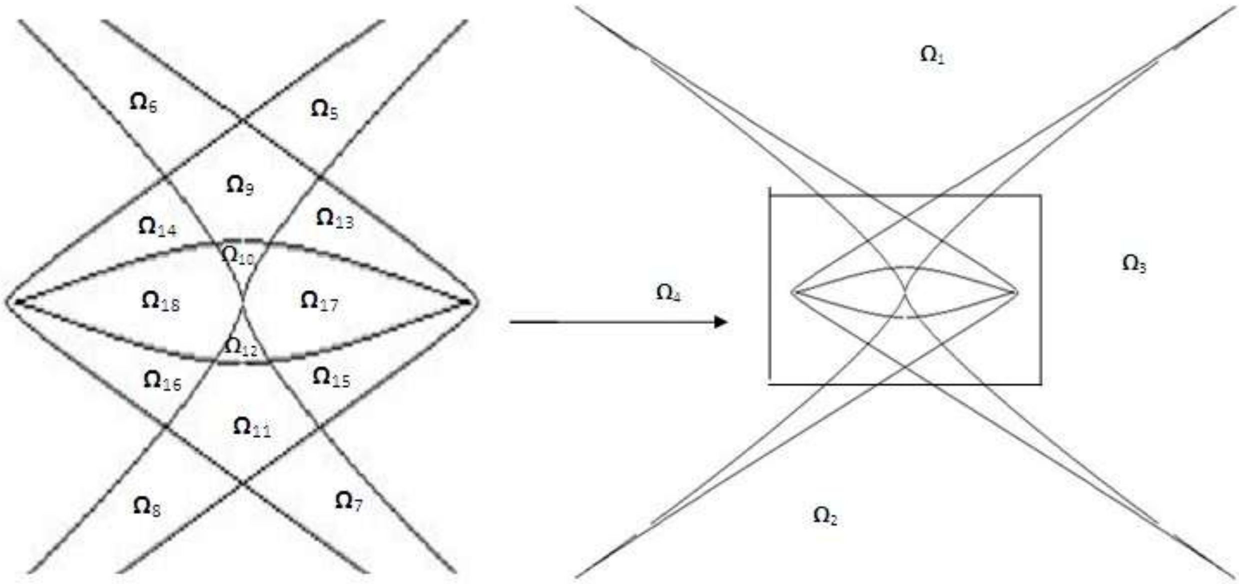


Fig.2 describe Caustic when $q_1 = -5$, $q_2 = -2.5$ and $k_1 = 0$

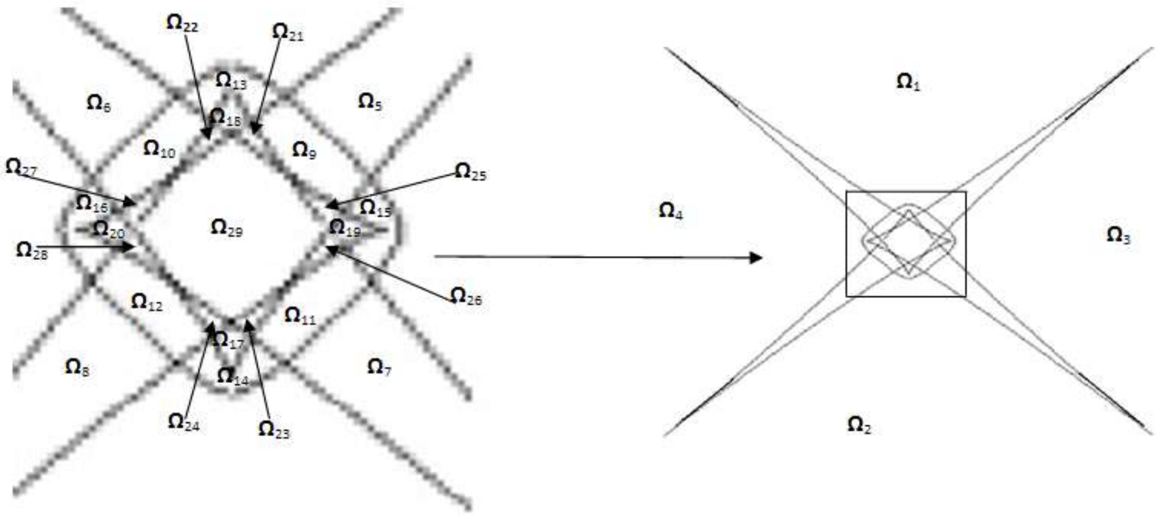


Fig.3 describe Caustic when $q_1 = -1$, $q_2 = -1$ and $k_1 = 0$

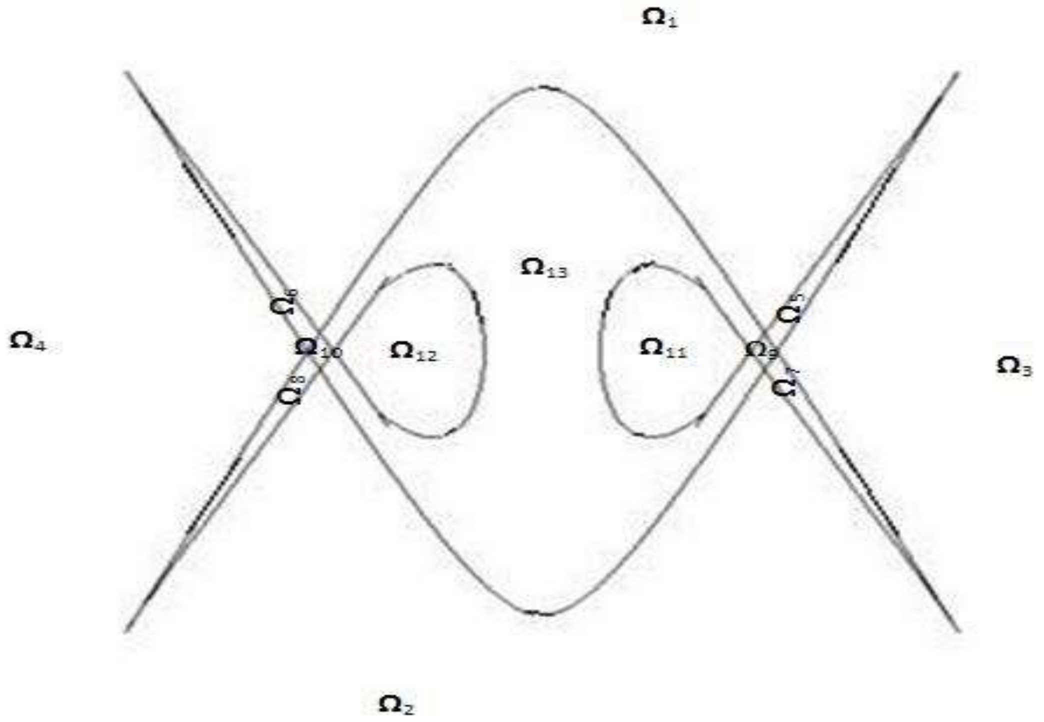
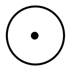
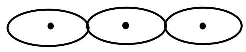
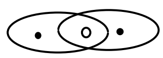


Fig.4 describe Caustic when $q_1 = 1.2,\ q_2 = -9.5\ and\ k_1 = 0$

The bifurcation propagation of the critical points of the function (2.2) is given as follows:

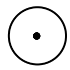
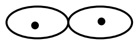
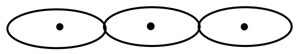
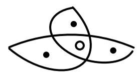
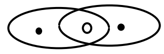
In fig.1 the Caustic decomposed the space of parameters into (15) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2, Ω_3 and Ω_4 there is one critical point (minimum), in regions Ω_5, Ω_6, Ω_7, Ω_8, Ω_{13} and Ω_{14} there is 3 critical points (2 min. and 1 saddle), in regions Ω_9 and Ω_{10} there is 5 critical points (3 min. and 2 saddle), in region Ω_{15} there is 5 critical points (2 min., 2 saddle and 1 max.). In the following table we summarize the rustles given above as well as we give the level curves of function (2.2) correspond to each region.

Table (2.1)

No.	the region	No of critical point	Type of critical point	Level curve
1	$\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4$	1	min.	
2	$\Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8,\ \Omega_{13},\ \Omega_{14}$	3	2 min. and 1 saddle	
3	$\Omega_9,\ \Omega_{10}$	5	3 min. and 2 saddle	
4	Ω_{15}	5	2 min., 2 saddle and 1 max.	

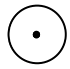
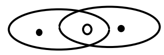
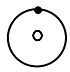
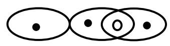
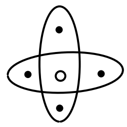
In fig.2 the Caustic decomposed the space of parameters into (18) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2, Ω_3 and Ω_4 there is one critical point (minimum), in regions Ω_5, Ω_6, Ω_7, Ω_8, Ω_{13}, Ω_{14}, Ω_{15} and Ω_{16} there is 3 critical points (2 min. and 1 saddle), in regions Ω_9 and Ω_{11} there is 5 critical points (2 symmetric min., 2 symmetric saddle and 1 min.), in regions Ω_{10} and Ω_{12} there is 7 critical points (2 symmetric min., 2 symmetric saddle ,1 min.,1 saddle and 1 max.), in regions Ω_{17} and Ω_{18} there is 5 critical points (2 min., 2 symmetric saddle and 1 max.). In the following table we summarize the rustles given above as well as we give the level curves of function (2.2) correspond to each region.

Table (2.2)

No.	the region	No of critical point	Type of critical point	Level curve
1	Ω_1, Ω_2, Ω_3, Ω_4	1	min.	
2	Ω_5, Ω_6, Ω_7, Ω_8, Ω_{13}, Ω_{14}, Ω_{15}, Ω	3	2 min. and 1 saddle	
3	Ω_9, Ω_{11}	5	2 symmetric min., 2 symmetric saddle and 1 min.	
4	Ω_{10}, Ω_{12}	7	2 symmetric min., 2 symmetric saddle ,1 min.,1 saddle and 1 max.	
5	Ω_{17}, Ω_{18}	5	2 min., 2 symmetric saddle and 1 max.	

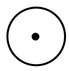
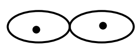
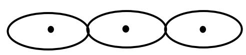
In fig.3 the Caustic decomposed the space of parameters into (29) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2, Ω_3 and Ω_4 there is one critical point (minimum), in regions Ω_9, Ω_{10}, Ω_{11}, Ω_{12}, Ω_{19} and Ω_{20} there is 5 critical points (2 min., 2 saddle and 1 max.), in regions Ω_{13}, Ω_{14}, Ω_{15}, and Ω_{16} there is 3 critical points (1 min.,1 saddle and 1 max.), in regions Ω_{17} and Ω_{18} there is 5 critical points (2 min., 2 symmetric saddle and 1 max.), in regions
Ω_{21}, Ω_{22}, Ω_{23}, Ω_{24}, Ω_{24}, Ω_{25}, Ω_{26} and Ω_{28} there is 7 critical points (3 min., 3 saddle and 1 max.), In regions Ω_{29} there is 9 critical points (4 min., 4 saddle and 1 max.). In the following table we summarize the rustles given above as well as we give the level curves of function (2.2) correspond to each region.

Table (2.3)

No.	the region	No of critical point	Type of critical point	Level curve
1	$\Omega_1, \Omega_2, \Omega_3, \Omega_4$	1	min.	
2	$\Omega_5, \Omega_6, \Omega_7, \Omega_8$	3	2 min. and 1 saddle	
3	$\Omega_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{19}, \Omega_{20}$	5	2 min., 2 saddle and 1 max.	
4	$\Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16}$	3	1 min.,1 saddle and 1 max.	
5	Ω_{17}, Ω_{18}	5	2 min., 2 symmetric saddle and 1 max.	
6	$\Omega_{21}, \Omega_{22}, \Omega_{23}, \Omega_{24}, \Omega_{24}, \Omega_{25}, \Omega_{26}, \Omega_{28}$	7	3 min., 3 saddle and 1 max.	
7	Ω_{29}	9	4 min., 4 saddle and 1 max.	

In fig.4 the Caustic decomposed the space of parameters into (13) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2, Ω_3 and Ω_4 there is one critical point (minimum), in regions Ω_5, Ω_6, Ω_7, Ω_8, Ω_{11}, Ω_{12} and Ω_{13} there is 3 critical points (2 min. and 1 saddle), in regions Ω_9 and Ω_{10} there is 5 critical points (3 min. and 2 saddle).

Table (2.4)

No.	the region	No of critical point	Type of critical point	Level curve
1	$\Omega_1, \Omega_2, \Omega_3, \Omega_4$	1	min.	
2	$\Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_{11}, \Omega_{12}, \Omega_{13}$	3	2 min. and 1 saddle	
3	Ω_9, Ω_{10}	5	3 min. and 2 saddle	

Case II:

When $a_4 = 0$ the function (2.1) can be written in the form of,

$$W_2(x,y,\lambda) = \frac{x^4}{4} + \frac{y^4}{4} + k_1x^2y^2 + k_2x^2y + k_3xy^2 + k_4x^2 + k_5y^2 - q_1x - q_2y \quad \text{... (2.4)}$$

where $a_4 = k_1$, $a_2 = k_2$, $a_3 = k_3$, $a_5 = k_4$, $a_6 = k_5$, $a_7 = -q_1$ and $a_8 = -q_2$

As in the previous case the critical points of the function (2.4) are the solutions of the following system,

$$x^3 + 2k_1xy^2 + 2k_2xy + k_3y^2 + 2k_4x - q_1 = 0$$
$$y^3 + 2k_1x^2y + k_2x^2 + 2k_3xy + 2k_5y - q_2 = 0.$$

and the caustic can be find by make the following parameterization

$$q_1 = x^3 + 2k_1xy^2 + 2k_2xy + k_3y^2 + 2k_4x$$
$$q_2 = y^3 + 2k_1x^2y + k_2x^2 + 2k_3xy + 2k_5y.$$

hence, for some values of k_1, k_2, k_3, k_4 and k_5 we have the following figures of the Caustic of function (2.4) in q_1q_2-plane,

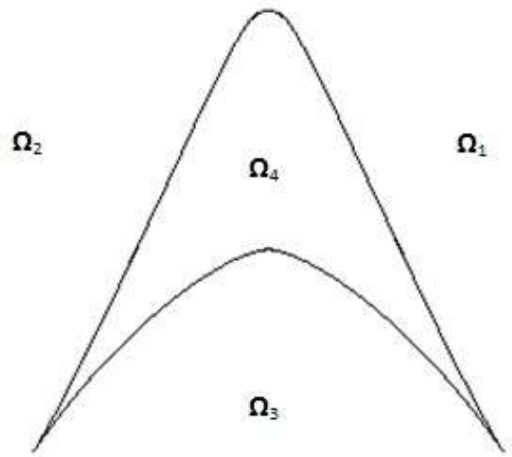


Fig.5 describe Caustic when $k_1 = k_4 = -.45,\ k_2 = k_5 = 0\ and\ k_3 = .45$

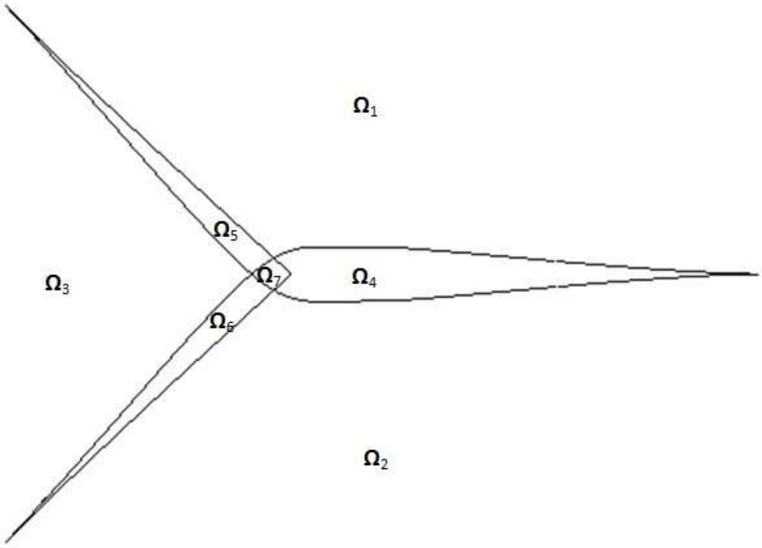
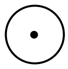
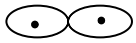


Fig.6 describe Caustic when $k_1 = -15,\ -10 \le k_2 \le 57,\ k_3 = .625, k_4 = 0\ \ and\ k_5 = -10$

The bifurcation propagation of the critical points of the function (2.4) is given as follows:

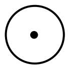
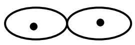
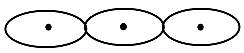
In fig.5 the Caustic decomposed the space of parameters into (4) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2 and Ω_3 there is one critical point (minimum), in region Ω_4 there is 3 critical points (2 min. and 1 saddle).

Table (2.5)

No.	the region	No of critical point	Type of critical point	Level curve
1	$\Omega_1,\ \Omega_2,\ \Omega_3$	1	min.	
2	Ω_4	3	2 min. and 1 saddle	

In fig.6 the Caustic decomposed the space of parameters into (13) regions, every region contains a fixed number of critical points. In regions Ω_1, Ω_2 and Ω_3 there is one critical point (minimum), in regions Ω_4, Ω_5 and Ω_6 there is 3 critical points (2 min. and 1 saddle), in region Ω_7 there is 5 critical points (3 min. and 2 saddle).

Table (2.6)

No.	the region	No of critical point	Type of critical point	Level curve
1	$\Omega_1,\ \Omega_2, \Omega_3$	1	min.	
2	$\Omega_4,\ \Omega_5, \Omega_6$	3	2 min. and 1 saddle	
3	Ω_7	5	3 min. and 2 saddle	

3. Classification of the Critical Points for Functions of Codimension Fifteen.

If $V_0(x,y) = \frac{x^6}{6} + \frac{y^4}{4}$, then the ideal generated by the partial derivatives of V_0 is given by

$$\left(\frac{\partial V_0}{\partial x}, \frac{\partial V_0}{\partial y}\right) = (x^5, y^3)$$

hence, the function V_0 has multiplicity equal to fifteen and codimension equal to fourteen. The goal of this section is to study the following function

$$V_0(x,y,\lambda) = \frac{x^6}{6} + \frac{y^4}{4} + x^4y^2 + k_1x^2 + k_2y^2 + k_3x + k_4y \qquad (3.1)$$

The critical points of the function (3.2) satisfying the following system of nonlinear algebraic equation,

$$x^5 + 4x^3y^2 + 2k_1x + k_3 = 0$$
$$y^3 + 2x^2y + 2k_2y + k_4 = 0.$$

and are degenerate on the surface given by the equation,

$$10x^8 - 40x^6y^2 + 15x^4\mathrm{y}^2 + 36x^2\mathrm{y}^4 + (4\mathrm{k}_1 + 10k_2)x^4 + 24k_2\mathrm{x}^2\mathrm{y}^2 + 6\mathrm{k}_1\mathrm{y}^2 + 4\mathrm{k}_1\mathrm{k}_2 = 0 \qquad . \quad (3.2)$$

To gate the Caustic of the function (3.2) we make the following parameterization,

$$k_3 = -(x^5 + 4x^3y^2 + 2k_1x)$$
$$k_4 = -(y^3 + 2x^2y + 2k_2y).$$

by changing the values of k_1 and k_2 we have the following geometric description of the Caustic,

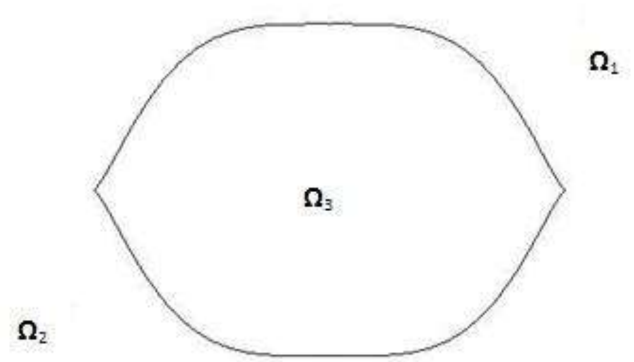


Fig.7 describe Caustic when $k_1 = 100 \quad and -3 \leq k_2 < 0$
Or $\quad k_1 \geq 85 \quad and \ k_2 = -2$

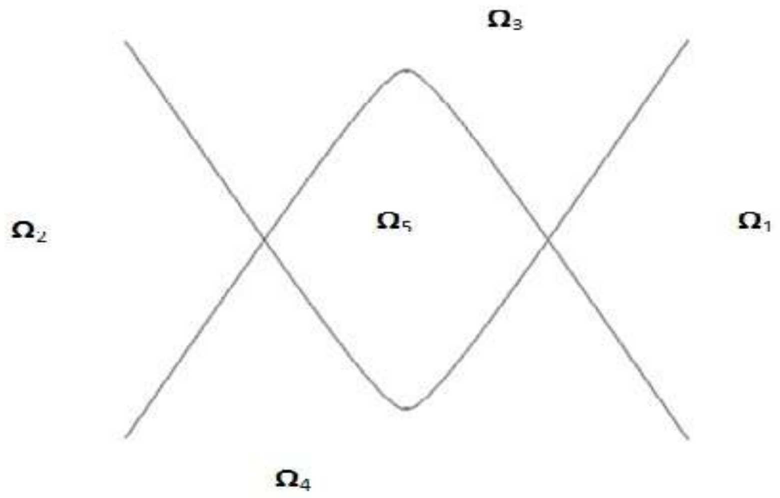
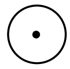
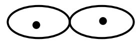


Fig.8 describe Caustic when $k_1 \geq 0$ and $\mathrm{k}_2 = -20$

The bifurcation spreading of the critical points of the function (3.2) is given as follows:

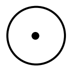
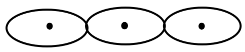
In fig.7 the Caustic decomposed the space of parameters into (3) regions, every region contains a fixed number of critical points. In regions Ω_1 and Ω_2 there is one critical point (minimum), in region $\boldsymbol{\Omega_3}$ there is 3 critical points (2 min. and 1 saddle).

Table (3.1)

No.	the region	No of critical point	Type of critical point	Level curve
1	Ω_1, Ω_2	1	min.	
2	Ω_3	3	2 min. and 1 saddle	

In fig.8 the Caustic decomposed the space of parameters into (5) regions, every region contains a fixed number of critical points. In regions Ω_1 and Ω_2 there is one critical point (minimum), in regions Ω_3 and Ω_4 there is 3 critical points (2 min. and 1 saddle), in region Ω_5 there is 5 critical points (3 min. and 2 saddle).

Table (3.2)

No.	the region	No of critical point	Type of critical point	Level curve
1	Ω_1, Ω_2	1	min.	
2	Ω_3, Ω_4	3	2 min. and 1 saddle	
3	Ω_5, Ω_6	5	3 min. and 2 saddle	

Conclusions

In this work we studied the classifications of critical points of some functions that appear as a reduced functions of the certain nonlinear Fredholm functionals. A new geometric description of the caustic we found, also we found the possible kinds of the level curves for each function. Two different cases we studied for the function (2.1), the first when $a_2 = a_3 = 0$ and the second when $a_4 = 0$. The function in section (2) has nine critical points while the function in section (3) has fifteen critical points. There are some similarities between the types of the critical points and the level curves of both functions, especially in the types of 1,3 and 5 critical points.

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